Advanced Systems Lab

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Lecture: Optimizing FFT, FFTW

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Fast FFT: Example FFTW Library

www.fftw.org

Frigo and Johnson, FFTW: An Adaptive Software Architecture for the FFT, ICASSP 1998

Frigo, A Fast Fourier Transform Compiler, PLDI 1999

Frigo and Johnson, The Design and Implementation of FFTW3, Proc. IEEE 93(2) 2005

Transform Algorithms

An algorithm for y = Tx is given by a factorization

$$T = T_1 T_2 \cdots T_m$$

Namely, instead of y = Tx we can compute in steps

$$\left.\begin{aligned} t_1 &= T_m x \\ t_2 &= T_{m-1} t_1 \\ &\cdots \\ y &= T_1 t_{m-1} \end{aligned}\right\} \text{m steps}$$

This reduces the op count only if:
- the T_i are sparse
- m is not too large

Example: Cooley-Tukey Fast Fourier Transform (FFT), size 4

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} x = \begin{bmatrix} 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & 1 \\ 1 & \cdot & -1 & \cdot \\ \cdot & 1 & \cdot & -1 \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & i \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdot & \cdot \\ 1 & -1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} x$$

12 adds
4 mults by i

4 adds

1 mult by i

4 adds

0 ops

Cooley-Tukey FFT, n = 4

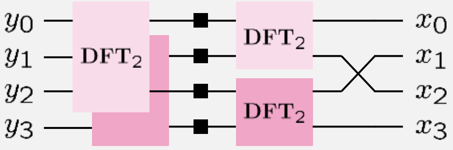
Fast Fourier transform (FFT)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} = \begin{bmatrix} 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & 1 \\ 1 & \cdot & -1 & \cdot \\ \cdot & 1 & \cdot & -1 \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & i \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdot & \cdot \\ 1 & -1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

Representation using matrix algebra

$$\mathbf{DFT}_4 = (\mathbf{DFT}_2 \otimes \mathrm{I}_2)\,\mathrm{diag}(1,1,1,i)(\mathrm{I}_2 \otimes \mathbf{DFT}_2)\,\mathrm{L}^4_2$$

Data flow graph (right to left)



stride 2 → stride 1

2 DFTs of size 2
at stride 2

2 DFTs of size 2

Recursive Cooley-Tukey FFT

radix

$$\mathbf{DFT}_{km} = (\mathbf{DFT}_{\overset{\downarrow}{k}} \otimes \mathrm{I}_m) T^{km}_m (\mathrm{I}_k \otimes \mathbf{DFT}_m) L^{km}_k$$ *decimation-in-time*

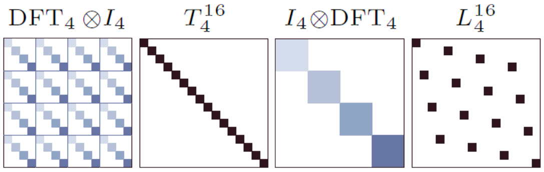
$$\mathbf{DFT}_{km} = L^{km}_m (\mathrm{I}_k \otimes \mathbf{DFT}_m) T^{km}_m (\mathbf{DFT}_k \otimes \mathrm{I}_m)$$ *decimation-in-frequency*

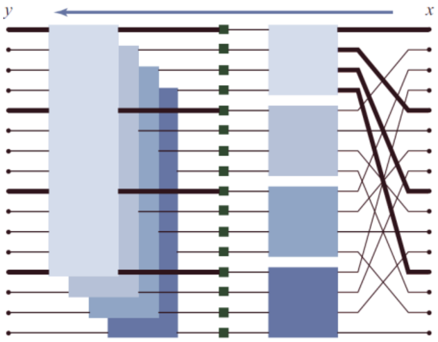
For powers of two n = 2^t sufficient together with base case

$$\mathbf{DFT}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

FFT, n = 16 *(Recursive, Radix 4)*

$$\mathrm{DFT}_{16} =$$





Fast Implementation (≈ FFTW 2.x)

Choice of algorithm

Locality optimization

Constants

Fast basic blocks

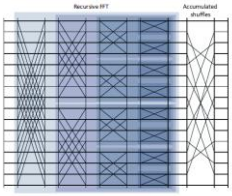
Adaptivity

1: Choice of Algorithm

Choose recursive, not iterative

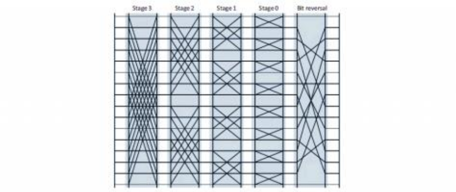
$$\mathbf{DFT}_{km} = (\mathbf{DFT}_k \otimes \mathrm{I}_m) T_m^{km} (\mathrm{I}_k \otimes \mathbf{DFT}_m) L_k^{km}$$

Radix 2, recursive



$(\mathrm{DFT}_2 \otimes I_8) T_8^{16} \Big(I_2 \otimes \Big((\mathrm{DFT}_2 \otimes I_4) T_4^8 (I_2 \otimes ((\mathrm{DFT}_2 \otimes I_2) T_2^4 (I_2 \otimes \mathrm{DFT}_2) L_2^4)) L_2^8 \Big) \Big) L_2^{16}$

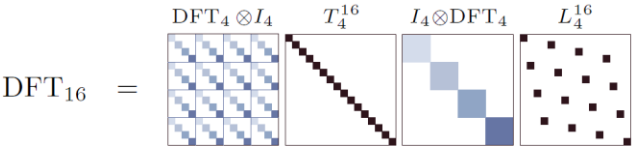
Radix 2, iterative



$\Big((I_1 \otimes \mathrm{DFT}_2 \otimes I_8) D_0^{16} \Big) \Big((I_2 \otimes \mathrm{DFT}_2 \otimes I_4) D_1^{16} \Big) \Big((I_4 \otimes \mathrm{DFT}_2 \otimes I_2) D_2^{16} \Big) \Big((I_8 \otimes \mathrm{DFT}_2 \otimes I_1) D_3^{16} \Big) R_2^{16}$

First recursive implementation we consider in this course

2: Locality Improvement



Straightforward implementation: 4 steps

- *Permute*
- *Loop recursively calling smaller DFTs (here: 4 of size 4)*
- *Loop that scales by twiddle factors (diagonal elements of T)*
- *Loop recursively calling smaller DFTs (here: 4 of size 4)*

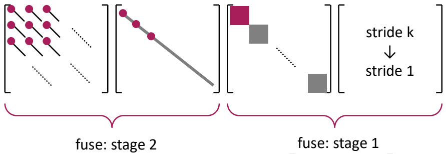
4 passes through data: bad locality

Better: fuse some steps

2: Locality Improvement

$$\mathrm{DFT}_n = (\mathrm{DFT}_k \otimes \mathrm{I}_m)\, \mathrm{T}^n_m (\mathrm{I}_k \otimes \mathrm{DFT}_m)\, \mathrm{L}^n_k$$

schematic:



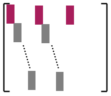
- compute m many DFT_k*D with input stride m and output stride m
- D is part of the diagonal T
- writes to the same location then it reads from → can be done in-place

Interface needed for recursive call:

`DFTscaled(k, x, d, m);`

k: DFT size
x: input = output vector
d: diagonal elements
m: input stride = output stride

Cannot handle further recursion so in FFTW it is a base case of the recursion



- compute k many DFT_m with input stride k and output stride 1
- writes to different location then it reads from → needs to be out-of-place

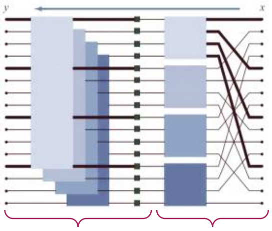
Interface needed for recursive call:

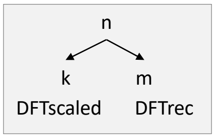
`DFTrec(m, x, y, k, 1);`

m: DFT size
x, y: input/ output vector
k: input stride
1: output stride

Can handle further recursion (just strides change)

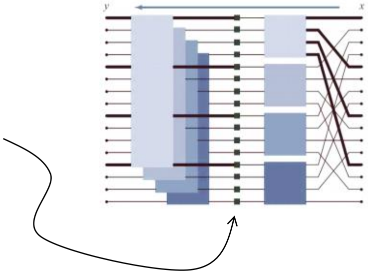
$$\mathrm{DFT}_{km} = \underbrace{(\mathrm{DFT}_k \otimes \mathrm{I}_m)T^{km}_m}_{\textit{one loop}}\underbrace{(\mathrm{I}_k \otimes \mathrm{DFT}_m)L^{km}_k}_{\textit{one loop}}$$





```
// code sketch
void DFT(int n, cpx *x, cpx *y) {
  …
  int k = choose_dft_radix(n); // ensure k small enough
  int m = n/k;
  for (int i = 0; i < k; ++i)
    DFTrec(m, x + i, y + m*i, k, 1); // implemented as DFT(…) is
  for (int j = 0; j < m; ++j)
    DFTscaled(k, y + j, t[j], m); // always a base case
}
```

3: Constants



FFT incurs multiplications by roots of unity

In real arithmetic:
Multiplications by sines and cosines, e.g.,

```
y[i] = sin(i·pi/128)*x[i];
```

Very expensive!

Observation: Constants depend only on input size, not on input

Solution: Precompute once and use many times

```
d = DFT_init(1024); // init function computes constant table
d(x, y);            // use many times
```

4: Optimized Basic Blocks

```
// code sketch
void DFT(int n, cpx *x, cpx *y) {
  if (use_base_case(n))
    DFTbc(n, x, y); // use base case
  else {
    int k = choose_dft_radix(n); // ensure k <= 32
    int m = n/k;
    for (int i = 0; i < k; ++i)
      DFTrec(m, x + i, y + m*i, k, 1); // implemented as DFT(…) is
    for (int j = 0; j < m; ++j)
      DFTscaled(k, y + j, t[j], m); // always a base case
  }
}
```

Just like loops can be unrolled, recursions can also be unrolled

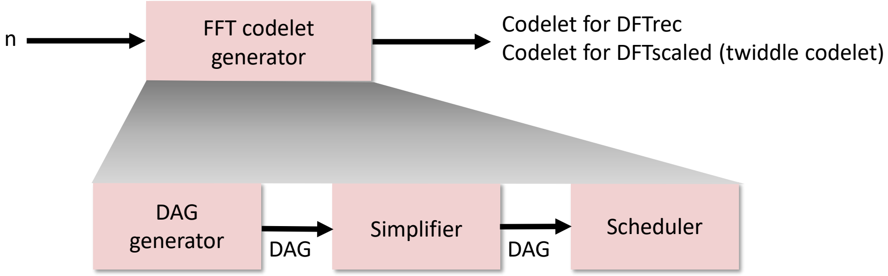
Empirical study: Base cases for sizes n ≤ 32 useful (scalar code)

Needs 62 base cases or “codelets” (why?)

- *DFTrec, sizes 2–32*
- *DFTscaled, sizes 2–32*

Solution: Codelet generator (codelet = optimized basic block)

FFTW Codelet Generator



n

FFT codelet generator

Codelet for DFTrec
Codelet for DFTscaled (twiddle codelet)

DAG generator

DAG

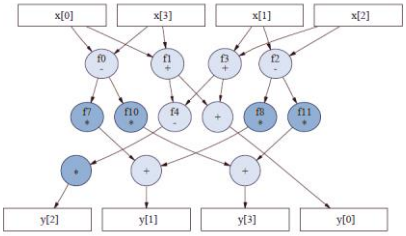
Simplifier

DAG

Scheduler

Small Example DAG

DAG:



One possible unparsing:

```
f0 = x[0] - x[3];
f1 = x[0] + x[3];
f2 = x[1] - x[2];
f3 = x[1] + x[2];
f4 = f1 - f3;
y[0] = f1 + f3;
y[2] = 0.7071067811865476 * f4;
f7 = 0.9238795325112867 * f0;
f8 = 0.3826834323650898 * f2;
y[1] = f7 + f8;
f10 = 0.3826834323650898 * f0;
f11 = (-0.9238795325112867) * f2;
y[3] = f10 + f11;
```

DAG Generator


DAG generator
DAG
Simplifier
DAG
Scheduler

Knows FFTs: Cooley-Tukey, split-radix, Good-Thomas, Rader, represented in sum notation

$$y_{n_2 j_1 + j_2} = \sum_{k_1=0}^{n_1-1} \left(\omega_n^{j_2 k_1}\right) \left(\sum_{k_2=0}^{n_2-1} x_{n_1 k_2 + k_1} \omega_{n_2}^{j_2 k_2}\right) \omega_{n_1}^{j_1 k_1}$$

For given n, suitable FFTs are recursively applied to yield n (real) expression trees for outputs y_0, …, y_{n-1}

Trees are fused to an (unoptimized) DAG

Simplifier

Applies:

- *Algebraic transformations*
- *Common subexpression elimination (CSE)*
- *DFT-specific optimizations*

Algebraic transformations

- *Simplify mults by 0, 1, -1*
- *Distributivity law: kx + ky = k(x + y), kx + lx = (k + l)x*
  Canonicalization: (x-y), (y-x) to (x-y), -(x-y)

CSE: standard

- *E.g., two occurrences of 2x+y: assign new temporary variable*

DFT specific optimizations

- *All numeric constants are made positive (reduces register pressure)*
- *CSE also on transposed DAG*

Scheduler

Determines in which sequence the DAG is unparsed to C
(topological sort of the DAG)
Goal: minimize register spills

A 2-power FFT has an operational intensity of I(n) = O(log(C)), where C is the cache size [1]

Implies: For R registers Ω(n log(n)/log(R)) register spills

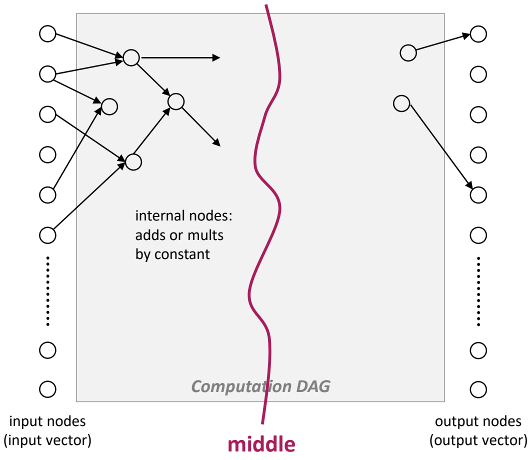
FFTW’s scheduler achieves this (asymptotic) bound *independent* of R

[1] Hong and Kung: “I/O Complexity: The red-blue pebbling game”

FFT-Specific Scheduler: Basic Idea

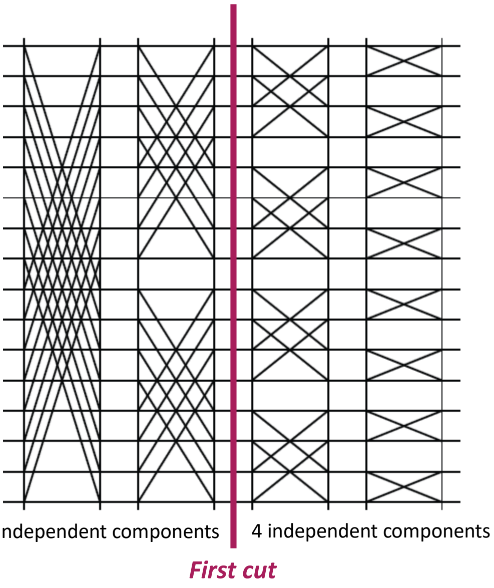
Cut DAG in the middle

Recurse on the connected components

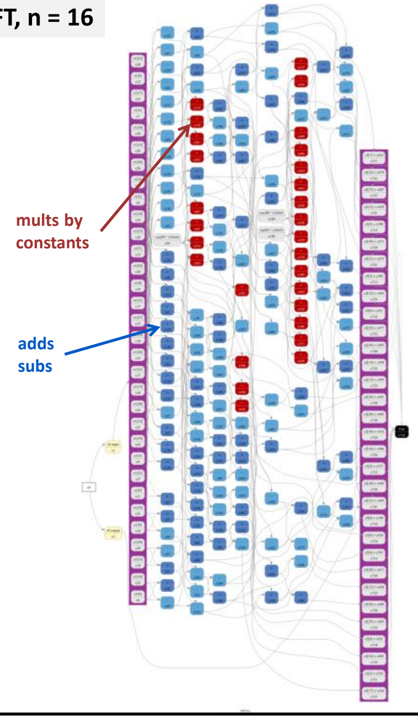


How to find the middle?

This is a Sketched/Abstracted DAG



FFT, n = 16



```
typedef struct {
                double* input;
                double* output;
} spiral_t;
const double x708[] = { 1.0, 0.9238795325112867, 0.70710678118654
const double x709[] = { -0.0, 0.3826834323650898, 0.7071067811865
void staged(spiral_t* x0) {
double* x2 = x0->output;
double* x1 = x0->input;
double x6 = x1[0];
double x22 = x1[16];
double x38 = x6 + x22;
double x14 = x1[8];
double x30 = x1[24];
double x46 = x14 + x30;
double x343 = x38 + x46;
double x10 = x1[4];
double x26 = x1[20];
double x42 = x10 + x26;
double x18 = x1[12];
double x34 = x1[28];
double x50 = x18 + x34;
double x344 = x42 + x50;
double x345 = x343 + x344;
double x8 = x1[2];
double x24 = x1[18];
double x115 = x8 + x24;
double x16 = x1[10];
double x32 = x1[26];
double x123 = x16 + x32;
double x346 = x115 + x123;
double x12 = x1[6];
double x28 = x1[22];
double x119 = x12 + x28;
double x20 = x1[14];
double x36 = x1[30];
double x127 = x20 + x36;
double x347 = x119 + x127;
double x348 = x346 + x347;
double x349 = x345 + x348;
x2[0] = x349;
double x7 = x1[1];
double x23 = x1[17];
double x39 = x7 + x23;
double x15 = x1[9];
double x31 = x1[25];
double x47 = x15 + x31;
double x76 = x39 + x47;
double x11 = x1[5];
double x27 = x1[21];
double x43 = x11 + x27;
double x19 = x1[13];
double x35 = x1[29];
double x51 = x19 + x35;
double x80 = x43 + x51;
double x88 = x76 + x80;
```

Codelet Examples

Notwiddle 2 (DFTrec)

Notwiddle 3 (DFTrec)

Twiddle 3 (DFTscaled)

Notwiddle 32 (DFTrec)

Code style:

- *Single static assignment (SSA)*
- *Scoping (limited scope where variables are defined)*

5: Adaptivity

Choices used for platform adaptation

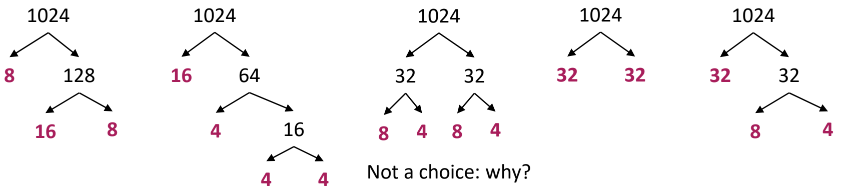
```
// code sketch
void DFT(int n, cpx *x, cpx *y) {
  if (use_base_case(n))
    DFTbc(n, x, y); // use base case
  else {
    int k = choose_dft_radix(n); // ensure k <= 32
    int m = n/k;
    for (int i = 0; i < k; ++i)
      DFTrec(m, x + i, y + m*i, k, 1); // implemented as DFT(…) is
    for (int j = 0; j < m; ++j)
      DFTscaled(k, y + j, t[j], m); // always a base case
  }
}
```

```
d = DFT_init(1024); // compute constant table; search for best recursion
d(x, y);            // use many times
```

5: Adaptivity

```
d = DFT_init(1024); // compute constant table; search for best recursion
d(x, y);            // use many times
```

Choices: $\mathbf{DFT}_{km} = (\mathbf{DFT}_k \otimes \mathrm{I}_m) T^{km}_m (\mathrm{I}_k \otimes \mathbf{DFT}_m) L^{km}_k$



Not a choice: why?

Base case = generated codelet is called

Exhaustive search to expensive

Solution: Dynamic programming

FFTW: Further Information

Previous Explanation: FFTW 2.x

FFTW 3.x:

- *Support for SIMD/threading*
- *Flexible interface to handle FFT variants (real/complex, strided access, sine/cosine transforms)*
- *Complicates significantly the interfaces actually used and increases the size of the search space*

	MMM ***Atlas***	**Sparse MVM** ***Sparsity/Bebop***	**DFT** ***FFTW***
Cache optimization			
Register optimization			
Optimized basic blocks			
Other optimizations			
Adaptivity			

	MMM *Atlas*	Sparse MVM *Sparsity/Bebop*	DFT *FFTW*
Cache optimization	Blocking	Blocking (rarely useful)	Recursive FFT, fusion of steps
Register optimization	Blocking	Blocking (changes sparse format)	Scheduling of small FFTs
Optimized basic blocks	Unrolling, scalar replacement and SSA, scheduling, simplifications (for FFT)		
Other optimizations	—	—	Precomputation of constants
Adaptivity	Search: blocking parameters	Search: register blocking size	Search: recursion strategy