Advanced Systems Lab

Spring 2021

Lecture: Discrete Fourier transform, fast Fourier transforms

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Linear Transforms

Overview: Transforms and algorithms

Discrete Fourier transform

Fast Fourier transforms

After that:

- *Optimized implementation and autotuning (FFTW)*
- *Automatic program synthesis (Spiral)*

FFT References

Complexity: *Bürgisser, Clausen, Shokrollahi, Algebraic Complexity Theory, Springer, 1997*

History: *Heideman, Johnson, Burrus: Gauss and the History of the Fast Fourier Transform, Arch. Hist. Sc. 34(3) 1985*

FFTs:

- *Cooley and Tukey,* ***An algorithm for the machine calculation of complex Fourier series****," Math. of Computation, vol. 19, pp. 297–301, 1965*
- *Nussbaumer,* ***Fast Fourier Transform and Convolution Algorithms****, 2nd ed., Springer, 1982*
- *van Loan,* ***Computational Frameworks for the Fast Fourier Transform****, SIAM, 1992*
- *Tolimieri, An, Lu,* ***Algorithms for Discrete Fourier Transforms and Convolution****, Springer, 2nd edition, 1997*
- *Franchetti, Püschel, Voronenko, Chellappa and Moura,* ***Discrete Fourier Transform on Multicore****, IEEE Signal Processing Magazine, special issue on ``Signal Processing on Platforms with Multiple Cores'', Vol. 26, No. 6, pp. 90-102, 2009*

Linear Transforms

Very important class of functions: signal processing, communication, scientific computing, …

Mathematically: Change of basis = Multiplication by a fixed matrix T

$$\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix} = y = Tx \longleftarrow \boxed{T\bullet} \longleftarrow x = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$$

$$T = [t_{k,\ell}]_{0 \le k,\ell < n}$$

Output ***Input***

Equivalent definition: Summation form

$$y_k = \sum_{\ell=0}^{n-1} t_{k,\ell} x_\ell, \quad 0 \le k < n$$

Linear Transforms

Compute: y = Tx x: input vector, y: output vector, T: fixed transform matrix

Example: Discrete Fourier transform (DFT)

1. form (standard in signal processing):

given: $x_0, \ldots, x_{n-1}$

compute:

$$y_k = \sum_{\ell=0}^{n-1} e^{-2k\ell\pi i/n} x_\ell, \quad k = 0, \ldots, n-1$$
$$= \sum_{\ell=0}^{n-1} \omega_n^{k\ell} x_\ell, \quad k = 0, \ldots, n-1, \quad \omega_n = e^{-2\pi i/n}$$

primitive nth root of 1

2. form (we will use):

given: $(x_0, \ldots, x_{n-1})^T$

compute: $y = \mathbf{DFT}_n \cdot x, \quad \mathbf{DFT}_n = [\omega_n^{k\ell}]_{0 \le k,\ell < n}$

How does the DFT_2 matrix look?
Second row of DFT_4 matrix?

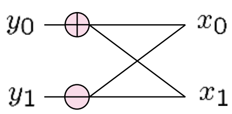
Smallest Relevant Example: DFT, Size 2

Transform (matrix): $\mathbf{DFT}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

How many ops to compute the DFT_2 of a vector?

Computation: $y = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} x$ ***or*** $\begin{aligned} y_0 &= x_0 + x_1 \\ y_1 &= x_0 - x_1 \end{aligned}$

As graph (direct acyclic graph or DAG):



called a butterfly

http://charlottesmartypants.blogspot.com/2011_02_01_archive.html

DFT, Size 4

$$\mathbf{DFT}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

How many (complex) operations to compute the DFT_4 of a (complex) vector?

$$y = \mathbf{DFT}_4 \cdot x$$

Transforms: Examples

A few dozen transforms are relevant

Some examples

$$\begin{aligned}
\mathbf{DFT}_n &= [e^{-2k\ell\pi i/n}]_{0\le k,\ell<n} \\
\mathbf{RDFT}_n &= [r_{k\ell}]_{0\le k,\ell<n}, \quad r_{k\ell} = \begin{cases} \cos\frac{2\pi k\ell}{n}, & k \le \lfloor\frac{n}{2}\rfloor \\ -\sin\frac{2\pi k\ell}{n}, & k > \lfloor\frac{n}{2}\rfloor \end{cases} \\
\mathbf{DHT} &= \left[\cos(2k\ell\pi/n) + \sin(2k\ell\pi/n)\right]_{0\le k,\ell<n} \\
\mathbf{WHT}_n &= \begin{bmatrix} \mathbf{WHT}_{n/2} & \mathbf{WHT}_{n/2} \\ \mathbf{WHT}_{n/2} & -\mathbf{WHT}_{n/2} \end{bmatrix}, \quad \mathbf{WHT}_2 = \mathbf{DFT}_2 \\
\mathbf{IMDCT}_n &= \left[\cos((2k+1)(2\ell+1+n)\pi/4n)\right]_{0\le k<2n, 0\le\ell<n} \\
\mathbf{DCT\text{-}2}_n &= \left[\cos(k(2\ell+1)\pi/2n)\right]_{0\le k,\ell<n} \\
\mathbf{DCT\text{-}3}_n &= \mathbf{DCT\text{-}2}_n^T \quad \text{(transpose)} \\
\mathbf{DCT\text{-}4}_n &= \left[\cos((2k+1)(2\ell+1)\pi/4n)\right]_{0\le k,\ell<n}
\end{aligned}$$

universal tool (DFT, RDFT)

MPEG (IMDCT)

JPEG (DCT-2)

Transform Algorithms

An algorithm for y = Tx is given by a factorization

$$T = T_1 T_2 \cdots T_m$$

Namely, instead of y = Tx we can compute in steps

$$\left.\begin{aligned} t_1 &= T_m x \\ t_2 &= T_{m-1} t_1 \\ &\dots \\ y &= T_1 t_{m-1} \end{aligned}\right\} \text{m steps}$$

This reduces the op count only if:
- the T_i are sparse
- m is not too large

Example: Cooley-Tukey Fast Fourier Transform (FFT), size 4

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} x = \begin{bmatrix} 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & 1 \\ 1 & \cdot & -1 & \cdot \\ \cdot & 1 & \cdot & -1 \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & i \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdot & \cdot \\ 1 & -1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} x$$

12 adds
4 mults by i

4 adds

1 mult by i

4 adds

0 ops

Cooley-Tukey FFT, n = 4

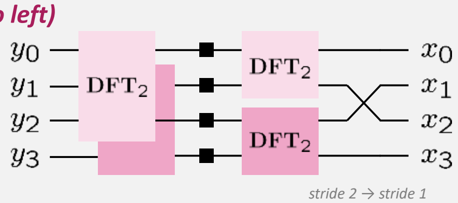
Fast Fourier transform (FFT)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} = \begin{bmatrix} 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & 1 \\ 1 & \cdot & -1 & \cdot \\ \cdot & 1 & \cdot & -1 \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & i \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdot & \cdot \\ 1 & -1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

Representation using matrix algebra

$$\mathbf{DFT}_4 = (\mathbf{DFT}_2 \otimes \mathrm{I}_2)\,\mathrm{diag}(1,1,1,i)(\mathrm{I}_2 \otimes \mathbf{DFT}_2)\,\mathrm{L}^4_2$$

Data flow graph (right to left)

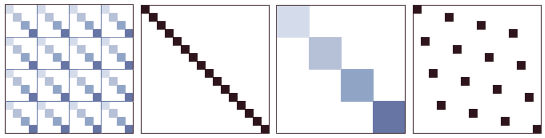


2 DFTs of size 2
at stride 2

2 DFTs of size 2

Example Recursive FFT, n = 16, radix 4

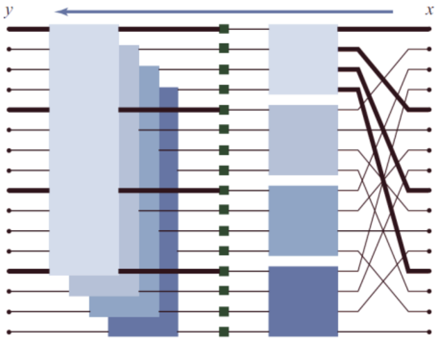
$$\mathrm{DFT}_{16} = (\mathrm{DFT}_4 \otimes I_4)\; T_4^{16}\; (I_4 \otimes \mathrm{DFT}_4)\; L_4^{16}$$



Kronecker product:

$A = [a_{ij}]_{i,j}$

$A \otimes B = [a_{ij} \cdot B]_{i,j}$



General Radix, Recursive Cooley-Tukey FFT

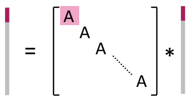
Assume n = km: $\mathbf{DFT}_n = (\mathbf{DFT}_k \otimes \mathrm{I}_m)\, \mathrm{T}_m^n (\mathrm{I}_k \otimes \mathbf{DFT}_m)\, \mathrm{L}_k^n$

permutation matrix

diagonal matrix with roots of unity

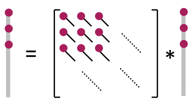
3 key structures: $\mathrm{I}_k \otimes A_m,\ A_k \otimes \mathrm{I}_m,\ \mathrm{L}_k^n$

$y = (\mathrm{I}_k \otimes A_m)x$




```
for i = 0:k-1
  y[im:im+m-1] = A*x[im:im+m-1]
```

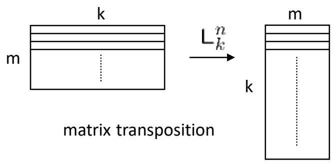
$y = (A_k \otimes \mathrm{I}_m)x$



m A's at stride m

```
for i = 0:m-1
  y[i:m:i+(k-1)m] = A*x[i:m:i+(k-1)m]
```

$y = \mathrm{L}_k^n x$ view x as m x k matrix:



matrix transposition

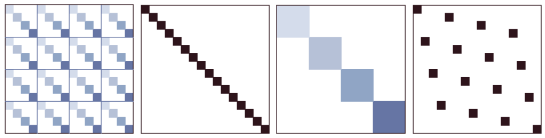
```
for i = 0:k-1
  for j = 0:m-1
    y[im+j] = x[i+kj]
```

read at stride k, write at stride 1

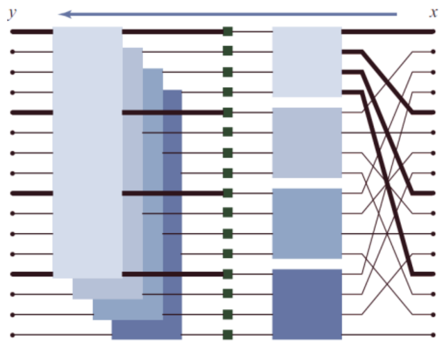
equivalent: read at stride 1, write at stride m

Example FFT, n = 16 *(Recursive, Radix 4)*

$$\mathrm{DFT}_{16} = (\mathrm{DFT}_4 \otimes I_4)\; T^{16}_4\; (I_4 \otimes \mathrm{DFT}_4)\; L^{16}_4$$



DFT$_4$ is expanded recursively



4 DFTs of size 4 at stride 4

4 DFTs of size 4

Recursive Cooley-Tukey FFT

$$\mathrm{DFT}_{km} = (\mathrm{DFT}_{\underset{\text{radix}}{k}} \otimes \mathrm{I}_m) T^{km}_m (\mathrm{I}_k \otimes \mathrm{DFT}_m) L^{km}_k \qquad \textit{decimation-in-time}$$

$$\mathrm{DFT}_{km} = L^{km}_m (\mathrm{I}_k \otimes \mathrm{DFT}_m) T^{km}_m (\mathrm{DFT}_k \otimes \mathrm{I}_m) \qquad \textit{decimation-in-frequency}$$

For powers of two n = 2^t sufficient together with base case

$$\mathrm{DFT}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Cost:

- *(complex adds, complex mults) = (n log$_2$(n), n log$_2$(n)/2)*
  independent of recursion
- *(real adds, real mults) ≤ (3n log$_2$(n), 2n log$_2$(n)) = 5n log$_2$(n) flops*
  depends on recursion: best is at least radix-8

Recursive vs. Iterative FFT

Recursive, radix-k Cooley-Tukey FFT

$$\mathbf{DFT}_{km} = (\mathbf{DFT}_k \otimes \mathrm{I}_m) T_m^{km} (\mathrm{I}_k \otimes \mathbf{DFT}_m) L_k^{km}$$

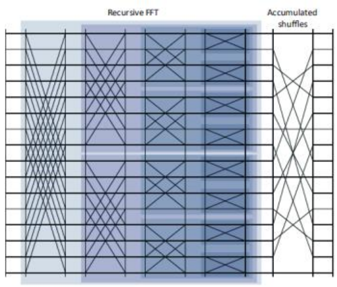
$$\mathbf{DFT}_{km} = L_m^{km} (\mathrm{I}_k \otimes \mathbf{DFT}_m) T_m^{km} (\mathbf{DFT}_k \otimes \mathrm{I}_m)$$

Iterative, radix 2, decimation-in-time/decimation-in-frequency

$$\mathbf{DFT}_{2^t} = \left(\prod_{j=1}^{t} (\mathrm{I}_{2^{j-1}} \otimes \mathbf{DFT}_2 \otimes \mathrm{I}_{2^{t-j}}) \cdot (\mathrm{I}_{2^{j-1}} \otimes T_{2^{t-j}}^{2^{t-j+1}}) \right) \cdot R_{2^t}$$

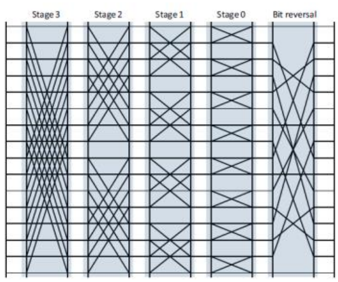
$$\mathbf{DFT}_{2^t} = R_{2^t} \cdot \left(\prod_{j=1}^{t} (\mathrm{I}_{2^{t-j}} \otimes T_{2^{j-1}}^{2^j}) \cdot (\mathrm{I}_{2^{t-j}} \otimes \mathbf{DFT}_2 \otimes \mathrm{I}_{2^{j-1}}) \right)$$

Radix 2, recursive



$$(\mathrm{DFT}_2 \otimes I_8) T_8^{16} \Big(I_2 \otimes \Big((\mathrm{DFT}_2 \otimes I_4) T_4^8 (I_2 \otimes ((\mathrm{DFT}_2 \otimes I_2) T_2^4 (I_2 \otimes \mathrm{DFT}_2) L_2^4)) L_2^8 \Big) \Big) L_2^{16}$$

Radix 2, iterative



$$\Big((I_1 \otimes \mathrm{DFT}_2 \otimes I_8) D_0^{16} \Big) \Big((I_2 \otimes \mathrm{DFT}_2 \otimes I_4) D_1^{16} \Big) \Big((I_4 \otimes \mathrm{DFT}_2 \otimes I_2) D_2^{16} \Big) \Big((I_8 \otimes \mathrm{DFT}_2 \otimes I_1) D_3^{16} \Big) R_2^{16}$$

Recursive vs. Iterative

Iterative FFT computes in stages of butterflies =
$\log_2(n)$ passes through the data

Recursive FFT reduces passes through data =
better locality

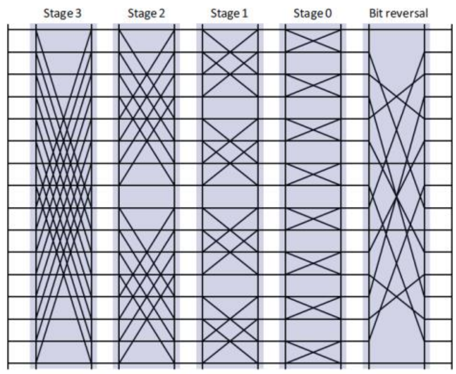
Same computation graph but different topological sorting

Rough analogy:

MMM	DFT
Triple loop	Iterative FFT
Blocked	Recursive FFT

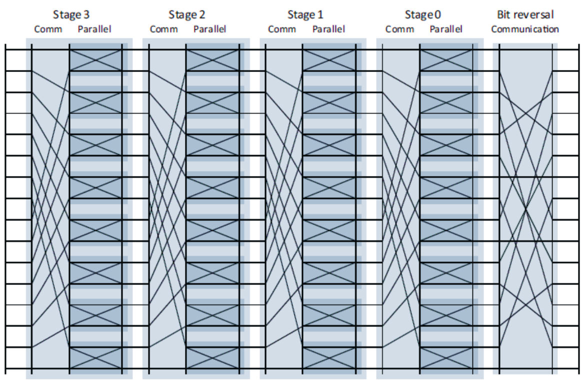
The FFT Is Very Malleable

Iterative FFT, Radix 2



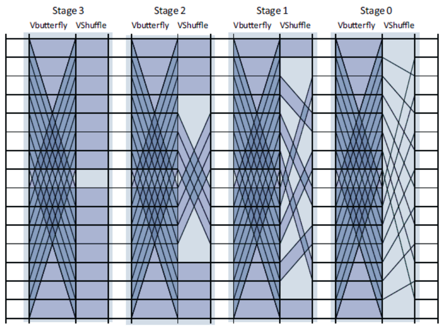
$$\left((I_1 \otimes \mathrm{DFT}_2 \otimes I_8) D_0^{16}\right)\left((I_2 \otimes \mathrm{DFT}_2 \otimes I_4) D_1^{16}\right)\left((I_4 \otimes \mathrm{DFT}_2 \otimes I_2) D_2^{16}\right)\left((I_8 \otimes \mathrm{DFT}_2 \otimes I_1) D_3^{16}\right) R_2^{16}$$

Pease FFT, Radix 2



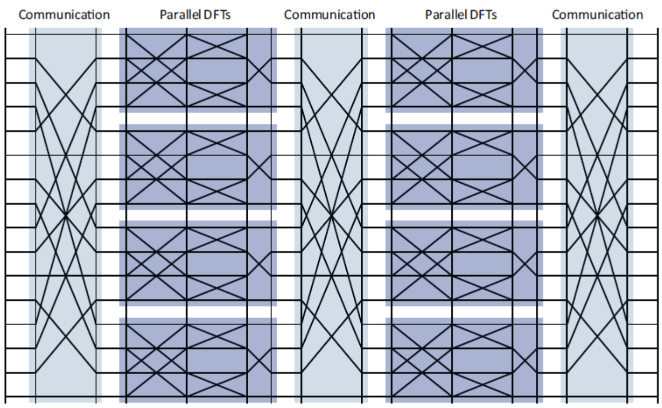
$$\left(L_2^{16}(I_8 \otimes \mathrm{DFT}_2) D_0^{16}\right)\left(L_2^{16}(I_8 \otimes \mathrm{DFT}_2) D_1^{16}\right)\left(L_2^{16}(I_8 \otimes \mathrm{DFT}_2) D_2^{16}\right)\left(L_2^{16}(I_8 \otimes \mathrm{DFT}_2) D_3^{16}\right) R_2^{16}$$

Stockham FFT, Radix 2



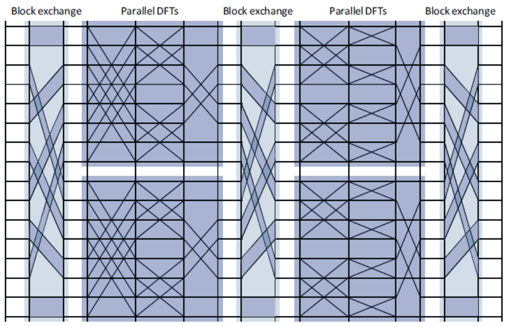
$$\Big((\mathrm{DFT}_2 \otimes I_8) D_0^{16} (L_2^2 \otimes I_8)\Big)\Big((\mathrm{DFT}_2 \otimes I_8) D_1^{16} (L_2^4 \otimes I_4)\Big)\Big((\mathrm{DFT}_2 \otimes I_8) D_2^{16} (L_2^8 \otimes I_2)\Big)\Big((\mathrm{DFT}_2 \otimes I_8) D_3^{16} (L_2^{16} \otimes I_1)\Big)$$

Six-Step FFT



$$L_4^{16}\Big(I_4 \otimes \big((\mathrm{DFT}_2 \otimes I_2) T_2^4 (I_2 \otimes \mathrm{DFT}_2) L_2^4\big)\Big) L_4^{16} T_4^{16}\Big(I_4 \otimes \big((\mathrm{DFT}_2 \otimes I_2) T_2^4 (I_2 \otimes \mathrm{DFT}_2) L_2^4\big)\Big) L_4^{16}$$

Multi-Core FFT



$$(L_4^8 \otimes I_2)\left(I_2 \otimes \left((\mathrm{DFT}_2 \otimes I_2) T_2^4 (I_2 \otimes \mathrm{DFT}_2) L_2^4\right) \otimes I_2\right)(L_2^8 \otimes I_2)\, T_4^{16} \left(I_2 \otimes \left(I_2 \otimes (\mathrm{DFT}_2 \otimes I_2) T_2^4 (I_2 \otimes \mathrm{DFT}_2)\right) R_2^8\right)(L_2^8 \otimes I_2)$$

Transform Algorithms

$$\mathbf{DFT}_n \rightarrow P_{k/2,2m}^{\top}\left(\mathbf{DFT}_{2m} \oplus \left(I_{k/2-1} \otimes_i C_{2m}\, \mathbf{rDFT}_{2m}(i/k)\right)\right)\left(\mathbf{RDFT}_k' \otimes I_m\right), \quad k \text{ even},$$

$$\left|\begin{matrix}\mathbf{RDFT}_n \\ \mathbf{RDFT}_n' \\ \mathbf{DHT}_n \\ \mathbf{DHT}_n'\end{matrix}\right| \rightarrow (P_{k/2,m}^{\top} \otimes I_2)\left(\left|\begin{matrix}\mathbf{RDFT}_{2m} \\ \mathbf{RDFT}_{2m}' \\ \mathbf{DHT}_{2m} \\ \mathbf{DHT}_{2m}'\end{matrix}\right| \oplus \left(I_{k/2-1} \otimes_i D_{2m} \left|\begin{matrix}\mathbf{rDFT}_{2m}(i/k) \\ \mathbf{rDFT}_{2m}(i/k) \\ \mathbf{rDHT}_{2m}(i/k) \\ \mathbf{rDHT}_{2m}(i/k)\end{matrix}\right|\right)\right)\left(\left|\begin{matrix}\mathbf{RDFT}_k' \\ \mathbf{RDFT}_k' \\ \mathbf{DHT}_k' \\ \mathbf{DHT}_k'\end{matrix}\right| \otimes I_m\right), \quad k \text{ even},$$

$$\left|\begin{matrix}\mathbf{rDFT}_{2n}(u) \\ \mathbf{rDHT}_{2n}(u)\end{matrix}\right| \rightarrow L_m^{2n}\left(I_k \otimes_i \left|\begin{matrix}\mathbf{rDFT}_{2m}((i+u)/k) \\ \mathbf{rDHT}_{2m}((i+u)/k)\end{matrix}\right|\right)\left(\left|\begin{matrix}\mathbf{rDFT}_{2k}(u) \\ \mathbf{rDHT}_{2k}(u)\end{matrix}\right| \otimes I_m\right),$$

$$\mathbf{RDFT\text{-}3}_n \rightarrow (Q_{k/2,m}^{\top} \otimes I_2)\,(I_k \otimes_i \mathbf{rDFT}_{2m})(i+1/2)/k))\,(\mathbf{RDFT\text{-}3}_k \otimes I_m), \quad k \text{ even},$$

$$\mathbf{DCT\text{-}2}_n \rightarrow P_{k/2,2m}^{\top}\left(\mathbf{DCT\text{-}2}_{2m} K_2^{2m} \oplus \left(I_{k/2-1} \otimes N_{2m}\, \mathbf{RDFT\text{-}3}_{2m}^{\top}\right)\right) B_n (L_{k/2}^{n/2} \otimes I_2)(I_m \otimes \mathbf{RDFT}_k') Q_{m/2,k},$$

$$\mathbf{DCT\text{-}3}_n \rightarrow \mathbf{DCT\text{-}2}_n^{\top},$$

$$\mathbf{DCT\text{-}4}_n \rightarrow Q_{k/2,2m}^{\top}\left(I_{k/2} \otimes N_{2m}\, \mathbf{RDFT\text{-}3}_{2m}^{\top}\right) B_n' (L_{k/2}^{n/2} \otimes I_2)(I_m \otimes \mathbf{RDFT\text{-}3}_k) Q_{m/2,k}.$$

$$\mathbf{DFT}_n \rightarrow (\mathbf{DFT}_k \otimes \mathrm{I}_m)\, \mathrm{T}_m^n (\mathrm{I}_k \otimes \mathbf{DFT}_m)\, \mathrm{L}_k^n, \quad n = km$$ *Cooley-Tukey FFT*

$$\mathbf{DFT}_n \rightarrow P_n (\mathbf{DFT}_k \otimes \mathbf{DFT}_m) Q_n, \quad n = km,\ \gcd(k,m) = 1$$ *Prime-factor FFT*

$$\mathbf{DFT}_p \rightarrow R_p^T (\mathrm{I}_1 \oplus \mathbf{DFT}_{p-1}) D_p (\mathrm{I}_1 \oplus \mathbf{DFT}_{p-1}) R_p, \quad p \text{ prime}$$ *Rader FFT*

$$\mathbf{DCT\text{-}3}_n \rightarrow (\mathrm{I}_m \oplus \mathrm{J}_m)\, \mathrm{L}_m^n (\mathbf{DCT\text{-}3}_m(1/4) \oplus \mathbf{DCT\text{-}3}_m(3/4)) \cdot (\mathrm{F}_2 \otimes \mathrm{I}_m) \begin{bmatrix} \mathrm{I}_m & 0 \oplus -\mathrm{J}_{m-1} \\ & \frac{1}{\sqrt{2}}(\mathrm{I}_1 \oplus 2\,\mathrm{I}_m) \end{bmatrix}, \quad n = 2m$$

$$\mathbf{DCT\text{-}4}_n \rightarrow S_n \mathbf{DCT\text{-}2}_n\, \mathrm{diag}_{0 \le k < n}(1/(2\cos((2k+1)\pi/4n)))$$

$$\mathbf{IMDCT}_{2m} \rightarrow (\mathrm{J}_m \oplus \mathrm{I}_m \oplus \mathrm{I}_m \oplus \mathrm{J}_m)\left(\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \otimes \mathrm{I}_m\right) \oplus \left(\begin{bmatrix} -1 \\ -1 \end{bmatrix} \otimes \mathrm{I}_m\right)\right) \mathrm{J}_{2m}\, \mathbf{DCT\text{-}4}_{2m}$$

$$\mathbf{WHT}_{2^k} \rightarrow \prod_{i=1}^{t} (\mathrm{I}_{2^{k_1+\cdots+k_{i-1}}} \otimes \mathbf{WHT}_{2^{k_i}} \otimes \mathrm{I}_{2^{k_{i+1}+\cdots+k_t}}), \quad k = k_1 + \cdots + k_t$$

$$\mathbf{DFT}_2 \rightarrow \mathrm{F}_2$$

$$\mathbf{DCT\text{-}2}_2 \rightarrow \mathrm{diag}(1, 1/\sqrt{2})\, \mathrm{F}_2$$

$$\mathbf{DCT\text{-}4}_2 \rightarrow \mathrm{J}_2\, \mathrm{R}_{13\pi/8}$$

Complexity of the DFT

Measure: L_c, $2 \le c$

- *Complex adds count 1*
- *Complex mults by a constant a with $|a| < c$ counts 1*
- *L_2 is strictest, L_∞ the loosest (and most natural)*

Upper bounds:

- *$n = 2^k$: $L_2(DFT_n) \le 3/2\, n \log_2(n)$ (using Cooley-Tukey FFT)*
- *General n: $L_2(DFT_n) \le 8\, n \log_2(n)$ (needs Bluestein FFT)*

Lower bound:

- *Theorem by Morgenstern: If $c < \infty$, then $L_c(DFT_n) \ge \frac{1}{2}\, n \log_c(n)$*
- *Implies: in the measure L_c, the DFT is $\Theta(n \log(n))$*

Lowest Known FFT Cost (Powers of 2)

A modified split-radix FFT with fewer arithmetic operations, *Johnson and Frigo, IEEE Trans. Signal Processing 55(1), pp. 111-119, 2007*

Number of flops ($n = 2^k$):

$$\frac{34}{9} n \log_2(n) - \frac{124}{27} n - 2 \log_2(n) - \frac{2}{9} (-1)^{\log_2(n)} \log_2(n) + \frac{16}{27} (-1)^{\log_2(n)} + 8$$

History of FFTs

The advent of digital signal processing is often attributed to the FFT
(Cooley-Tukey 1965)

History:

- *Around 1805: FFT discovered by Gauss [1]*
  (Fourier publishes the concept of Fourier analysis in 1807!)
- *1965: Rediscovered by Cooley-Tukey*

[1]: Heideman, Johnson, Burrus: "Gauss and the History of the Fast Fourier Transform" Arch. Hist. Sc. 34(3) 1985

Carl-Friedrich Gauss

1777 - 1855

Contender for the greatest mathematician of all times

Some contributions: Modular arithmetic, least square analysis, normal distribution, fundamental theorem of algebra, Gauss elimination, Gauss quadrature, Gauss-Seidel, non-Euclidean geometry, …