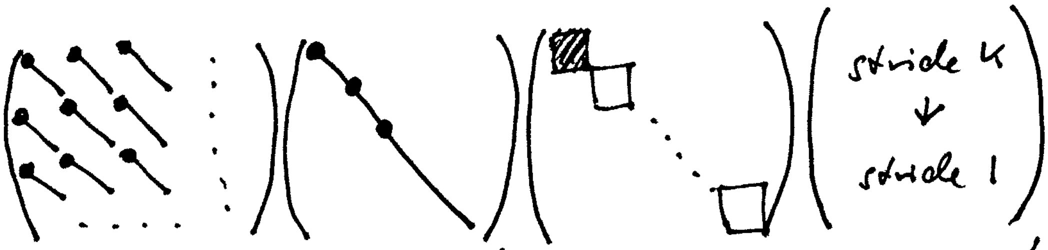
FFT, fast implementation (following FFTW 2.x)

1.) Choice of algorithm: Choose recursive FFT, not iterative FFT

2.) Locality optimization:

$$DFT_{km} = (DFT_k \otimes I_m)\, T^{km}_m\, (I_k \otimes DFT_m)\, L^{km}_k$$

(schematic) =



fuse (stage 2) — fuse (stage 1)

compute m many $DFT_k \cdot D$ at stride m (input and output)

D ↑ part of diagonal T^{km}_m

→ writes to the same location it reads from

→ inplace

DFTscaled(k, *x, *d, stride)

↑ k: size; ↑ x: input = output vector; ↑ d: diagonal elements

this interface cannot handle arbitrary recursions

→ in FFTW a base case

compute k many DFT_m with input stride k and output stride 1.

→ writes to different location it reads from

→ out-of-place

DFTrec(m, *x, *y, instride, outstride)

↑ m: size; ↑ x: input vector; ↑ y: output vector

this interface can handle arbitrary recursion

<u>Pseudocode</u>:

```
DFT(n, x, y) = DFTrec(n, x, y, 1, 1)
  for (int i = 0; i < k; ++i)
    DFTrec(m, y + m*i, x + i, k, 1); // implemented as DFT(…) is
  for (int j = 0; j < m; ++j)
    DFTscaled(k, y + j, t[j], m); // always a base case
```

t[j] ↑ precomputed twiddles

$$y \xleftarrow{\text{stage 2}} y \xleftarrow{\text{stage 1}} x$$

3.) Constants:

The matrix T_m^{km} yields multiplications by constants:

$$y_i = \underbrace{\omega_n^7}_{\text{some root of unity}} x_i$$

which in the code, on real numbers, gives multiplications by sines and cosines

$$y_i = \sin\left(\frac{i\pi}{128}\right) x_i \quad \text{etc....}$$

Problem: Computing $\sin(..)$ is very expensive (HW 2)

Solution:
- precompute once
- reuse many times
- assumes a transform for one size is used many times

Changes library interface:

```
d = dft-plan(1024);    // precomputes constants
d(*x, *y);             // computes DFT, size 1024
```

4.) Fast basic blocks

We do not want to recurse all the way to $n=2$
- function call overhead
- suboptimal register use

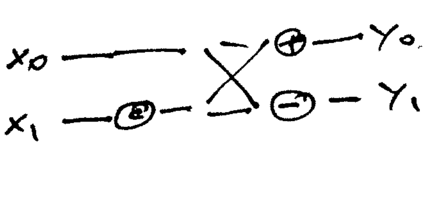
Solution:
- unroll recursion for small enough n
- practice shows $n \le 32$ is sufficient
- requires 62 functions! Why?

FFTW: "codelet" generator for small size FFT

$n \rightarrow$ [DAG generator] $\rightarrow$ [Simplifier] $\rightarrow$ [Scheduler] $\rightarrow$ straight line code for DFTrec(n.. or DFTscaled(n

a.) DAG generator
- generates DAG recursively from stored algorithms
- DAGs have only adds/subs/mults by const

Example:



$$\longleftrightarrow \quad \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & \\ & c \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

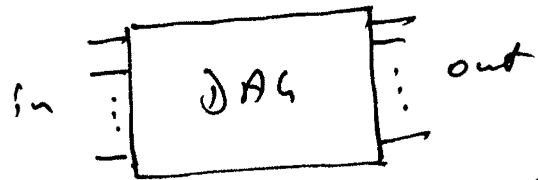
b.) Simplifier

- simplifies mults by 0, 1, -1
- distributivity law: $kx + ky = k(x+y)$
- canonicalization: $x-y, y-x \to x-y, -(x-y)$
- common subexpression elimination (CSE)
- all constants are made positive: reduces register pressure
- CSE also on transposed DAG

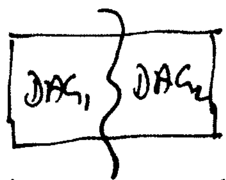
c.) Scheduler:

Theoretical result: 2-power FFT needs $\Omega\left(\frac{n\log(n)}{R}\right)$ register spills for R registers

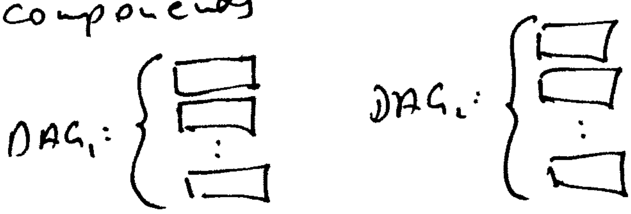
The following algorithm achieves that:



step 1: cut DAG in middle (how to do that



step 2: DAG_1, DAG_2 ~~...~~ decompose into independent components



schedule these recursively

Finally: output straightline, SSA code