General radix, recursive Cooley-Tukey FFT

assume $n = km$

$$DFT_{km} = (\underset{\substack{\uparrow \\ \text{radix}}}{DFT_k} \otimes I_m)\, \underbrace{T^n_m}_{\substack{\text{diagonal} \\ \text{matrix}}}\, (I_k \otimes DFT_m)\, L^n_k$$

3 key structures: $I_k \otimes A_m$, $A_k \otimes I_m$, L^n_k

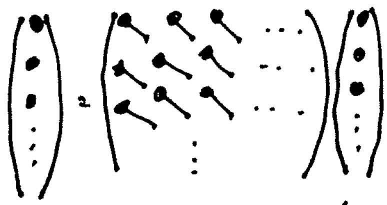
1.) $y = (I_k \otimes A_m)\, x$

$$\begin{pmatrix} \vdots \end{pmatrix} = \begin{pmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{pmatrix} \begin{pmatrix} \vdots \end{pmatrix}$$

k A's at stride 1

for $i = 0 : k-1$
 $y[im : 1 : im+m-1]$
$= A \cdot x[$ " $]$

2.) $y = (A_k \otimes I_m)\, x$



m A's at stride m

for $i = 0 : m-1$
 $y[i : m : i+(k-1)m]$
$= A \cdot x[$ " $]$

Permutes x to obtain y

3.) $y = L^n_k\, x$: different ways of viewing it

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for i = 0:k-1
  for j = 0:m-1 // m = n/k
    y[i*m+j] = x[i+k*j]
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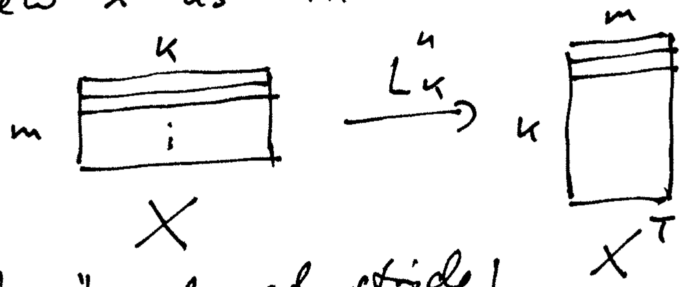
a.) view x as $m \times k$ matrix:



transposition!

b.) L^n_k "reads at stride 1 and writes at stride m"

stride 1 -> m is the same as stride k -> 1

c.) L^n_k performs permutation $im + j \to jk + i$ $0 \le i < k$, $0 \le j < m$

FFT again:

$$DFT_{km} = \underbrace{(DFT_k \otimes I_m)}_{\substack{\text{stride} \\ m \to m}}\, T^n_m\, \underbrace{(I_k \otimes DFT_m)}_{\substack{\text{stride} \\ 1 \to 1}}\, \underbrace{L^n_k}_{\substack{\text{stride} \\ 1 \to m}}$$

stride 1 -> m is the same as stride k -> 1

this is the "decimation-in-time" version

Decimation in frequency: transpose!

Use: - DFT is symmetric

- $(L^n_k)^T = L^n_m$

- $(A \otimes B)^T = A^T \otimes B^T$

Gives:

$$DFT_{km} = L^n_m (I_k \otimes DFT_m) T^n_m (DFT_k \otimes I_m)$$

Cost analysis: (was in exam for radix 2), assume $n = 2$

Measure: (complex adds, complex mults)

Cost: independent of radix $(n \log_2(n), \frac{1}{2} n \log_2(n))$

complex add = 2 real adds

" mult $\leq$ 4 real mults
2 real adds

$\Rightarrow$ real cost $\leq 2n \log_2(n) + 3n \log_2(n) = 5n \log_2(n)$

Iterative radix-2 FFT

$$DFT_{2^t} = \underbrace{R_{2^t}}_{\text{bit-reversal permutation}} \prod_{i=1}^{t} \underbrace{(I_{2^{t-i}} \otimes T^{2^i}_{2^{i-1}})}_{\text{diagonal matrix}} \underbrace{(I_{2^{t-i}} \otimes DFT_2 \otimes I_{2^{i-1}})}_{2^{t-1}\ DFT_2\text{'s at varying strides}}$$

Most people consider this "the FFT"