DIAGONALIZABLE SHIFT AND FILTERS FOR DIRECTED GRAPHS BASED ON THE JORDAN-CHEVALLEY DECOMPOSITION

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ABSTRACT

Graph signal processing on directed graphs poses theoretical challenges since an eigendecomposition of filters is in general not available. Instead, Fourier analysis requires a Jordan decomposition and the frequency response is given by the Jordan normal form, whose computation is numerically unstable for large sizes. In this paper, we propose to replace a given adjacency shift A by a diagonalizable shift A_D obtained via the Jordan-Chevalley decomposition. This means, as we show, that A_D generates the subalgebra of all diagonalizable filters and is itself a polynomial in A (i.e., a filter). For several synthetic and real-world graphs, we show how A_D adds and removes edges compared to A.

***Index Terms*—** graph signal processing, digraphs, Jordan normal form, algebraic signal processing, diagonalizable filters

1. INTRODUCTION

There is a plethora of data that is, or can be viewed as, indexed by the vertices of graphs. Examples include biological networks, social networks, or communication networks such as the internet [1, 2]. To bring signal processing (SP) tools to such graph data, fundamental SP concepts including shift, filters, Fourier transform, and frequency response, have been generalized to the graph domain [3, 4] and build the foundation of graph signal processing (GSP). There are two basic variants of GSP. The framework in [4] builds on algebraic signal processing (ASP) [5] to derive these concepts from the definition of the shift, given by the adjacency matrix. In contrast, [3] defines the eigenbasis of the graph Laplacian as the graph Fourier basis. In ASP terms, it chooses the Laplacian matrix as shift operator.

Undirected graphs. Both approaches yield a satisfying GSP framework for undirected graphs. Namely, since the shift operator is symmetric, a unitary Fourier basis exists. As a consequence, the shift, and thus all filters (polynomials in the shift) are diagonalizable, have one-dimensional frequency responses, and Parseval's theorem holds. Using these GSP tools many applications for graph signals have been developed, e.g., for compression, sampling, denoising, label propagation, outlier detection and alias-free filtering [4, 6, 7, 8]. In addition, graph convolutions are the foundation of graph convolutional neural networks that have been applied to supervised [9] and semisupervised learning tasks [10].

Directed graphs. Unfortunately, for directed graphs (digraphs), the GSP theory does not translate as well into practice. The reason is that the Fourier basis, given by subspaces that are invariant under filtering, is now determined by Jordan subspaces and the frequency response by the Jordan normal form. This results in various challenges for GSP theory and applications including:

1. Frequency components are no longer one-dimensional.
2. The Fourier basis and transform are not unitary.
3. The computation of the Jordan decomposition is numerically unstable [11, 12].

There have been various attempts to overcome these problems. Reference [13] replaces the Jordan basis with the basis corresponding to the block-diagonal Schur factorization, which factorizes a matrix A into a block-diagonal matrix $T = FAF^{-1}$. Similar to the Jordan basis, this basis decomposes the signal space into filtering invariant subspaces, but, not necessarily the irreducible ones. Reference [14] introduces a Hermitian Laplacian operator based on a generalization of the Hermitian adjacency matrix [15]. The Hermitian Laplacian is as the name suggests Hermitian and, by construction, captures the directions of the edges of the underlying graph. The work in [16, 17] defines the directed graph Fourier transform as the orthonormal basis with either minimal directed total variation or maximum spread, respectively. Further, [18] addresses the ambiguity in the choice of Jordan base vectors and proposes a basis-free computation of spectral components.

Contributions. In this work, we stay within the GSP framework of [4] and make use of the Jordan-Chevalley decomposition [19, 20] to derive a *diagonalizable shift* A_D from a given digraph shift (adjacency matrix) A. We show that A_D is a polynomial in A (i.e., a valid filter) and that it generates

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the set of all diagonalizable filters. More precisely, the diagonalizable polynomials in A are precisely the polynomials in A_D. We present prototypical experiments with synthetic and real-world graphs. They show that A_D often differs by a relatively small number of edges from A. This suggests that it might be possible to amend a graph given by A to A_D to overcome the problems with Jordan bases.

2. SIGNAL PROCESSING ON DIRECTED GRAPHS

We briefly review graph signal processing for directed graphs (digraphs) as introduced in [4]. Let G be a weighted digraph with vertices $V = \{v_1, \dots, v_n\}$, edges E, and an adjacency matrix $A \in \mathbb{C}^{n \times n}$ containing the weights of the edges.

Graph signal. A graph signal on G is a signal indexed by its vertices

$$s : V \to \mathbb{C}; v \mapsto s_v. \tag{1}$$

For mathematical convenience, we fix a vertex ordering and write the signal as column vector $\mathbf{s} = (s_{v_1}, \dots, s_{v_n})^T$.

Graph shift. GSP [4] is an instantiation of the algebraic signal processing theory [5] to graphs. Hence, convolution, filters and Fourier transform are derived from the definition of a graph shift (that we also denote with A):

$$A : \mathbb{C}^n \to \mathbb{C}^n;\ \mathbf{s} \mapsto A\mathbf{s}. \tag{2}$$

It is worth mentioning that the standard cyclic shift used for finite time series is the graph shift on the directed circle graph.

Filters. The corresponding graph filters are linear, shift invariant mappings given by polynomials in A of the form

$$H : \mathbb{C}^n \to \mathbb{C}^n; \mathbf{s} \mapsto \sum_{i=0}^{k} h_i A^i \mathbf{s}. \tag{3}$$

The matrix associated with H is $\sum_{i=0}^{k} h_i A^i$, which implies shift-invariance: $H(A\mathbf{s}) = AH(\mathbf{s})$. The set of all such filters is closed under polynomial addition and multiplication and thus forms an algebra $\mathcal{A}$. The filter algebra $\mathcal{A}$ is isomorphic to the polynomial algebra $\mathbb{C}[x]/m_A(x)$, where $m_A(x)$ denotes the minimal polynomial of A. We write the minimal polynomial of A as $m_A(x) = \prod_{i=1}^{k}(x - \lambda_i)^{d_i}$, where the λ_i denote A's distinct eigenvalues and the d_i the associated lengths of their longest Jordan chains.

Example 1. *The graph with adjacency matrix A in Fig. 1a has the minimal polynomial $x^3(x + \sqrt{2})(x - \sqrt{2})$.*

Fourier transform. Let $J = FAF^{-1}$ be the Jordan normal form (JNF) of A. Then F is the Fourier transform that decomposes the signal space $\mathbb{C}^n$ into a direct sum of the smallest subspaces that are invariant under the shift and thus all filters:

$$F :\ \mathbb{C}^n \to \bigoplus_{i=1}^{k} \bigoplus_{j=1}^{g_i} S_{ij};\ \mathbf{s} \mapsto F\mathbf{s}.$$

$$\begin{pmatrix} 0&0&2&0&0&1&0\\ 1&0&1&0&0&1&0\\ 1&0&0&0&0&0&0\\ 0&0&0&0&0&1&1\\ 0&0&1&0&0&1&1\\ 0&0&0&0&0&0&0\\ 1&0&1&0&0&1&0 \end{pmatrix} \quad \begin{pmatrix} 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&-\sqrt{2}&0\\ 0&0&0&0&0&0&\sqrt{2} \end{pmatrix} \quad \begin{pmatrix} 0&0&2&0&0&1&0\\ 1&0&1&0&0&\frac{1}{2}&0\\ 1&0&0&0&0&0&0\\ \frac{1}{2}&0&1&0&0&\frac{1}{2}&0\\ \frac{1}{2}&0&2&0&0&1&0\\ 0&0&0&0&0&0&0\\ 1&0&1&0&0&\frac{1}{2}&0 \end{pmatrix}$$

(a) A (b) $J = FAF^{-1}$ (c) $A_D = p(A)$

Fig. 1: (a) The adjacency matrix of our example, (b) the associated Jordan normal form of A, and (c) an associated diagonalizable shift derived in this paper.

Each S_{ij} denotes the subspace of $\mathbb{C}^n$ spanned by the Jordan chain corresponding to the j-th eigenvector for the i-th eigenvalue λ_i. The geometric multiplicity g_i is the number of such chains, i.e., the dimension of the eigenspace for λ_i. Invariance means that for $\mathbf{s} \in S_{ij}$ and $H \in \mathcal{A}$, we have $H\mathbf{s} \in S_{ij}$.

Frequency response. The frequency response of a filter $H = h(A)$ captures its action on the pure frequencies (= columns of F^{-1}). Thus, it is given by

$$FHF^{-1} = h(J). \tag{4}$$

Example 2. *The frequency response of the graph shift A in Fig. 1a) is given by its JNF in Fig. 1b.*

3. DIAGONALIZABLE DIGRAPH FILTERS

In this section we present our main contribution. For a given digraph shift A, we constructively derive an associated *diagonalizable shift* A_D. The new shift A_D is a polynomial in A (i.e., a filter) and generates the algebra of all diagonalizable filters. Further, as we see later in the experiments, if A_D is again interpreted as graph, it often differs from A by only a small number of edges.

3.1. Diagonalizable Digraph Shift

We use the Jordan-Chevalley decomposition of algebras [19, 20] imported to the GSP setting, i.e., algebras of the form $\mathbb{C}[x]/m_A(x)$ generated by a matrix A.

Theorem 1. (Jordan-Chevalley Decomposition) *A matrix $A \in \mathbb{C}^{n \times n}$ can be uniquely decomposed into the sum of two matrices $A = A_D + A_N$, with A_D and $A_N \in \mathbb{C}^{n \times n}$, that satisfy the following properties:*

1. *A_D is diagonalizable,*
2. *A_N is nilpotent (i.e., a suitable power is 0),*
3. *A_D and A_N commute, i.e., $A_D A_N = A_N A_D$,*
4. *A_D and A_N are polynomials in A, i.e., $A_D = p(A)$ and $A_N = A - p(A)$.*

To prove Theorem 1, we need the following lemma about the frequency response on a single Jordan subspace S_{ij}. The lemma was already used implicitly in [4, App. B & C].

(a) A (b) $A_D = A - A_N$

Fig. 2: (a) Directed graph with adjacency matrix A in Fig. 1a, (b) the graph corresponding to the diagonalizable A_D. Edges with weights $\neq 1$ are labelled, new edges are red.

Lemma 1. (Polynomial of Jordan block) *Let $J_d(\lambda)$ be a Jordan block of size d for eigenvalue λ. The polynomial $h \in \mathbb{C}[x]/m_A(x)$ evaluated at $J_d(\lambda)$ takes the form*

$$h(J_d(\lambda)) = \begin{pmatrix} h(\lambda) & \frac{h^{(1)}(\lambda)}{1!} & \cdots & \frac{h^{(d-1)}(\lambda)}{(d-1)!} \\ 0 & h(\lambda) & \cdots & \frac{h^{(d-2)}(\lambda)}{(d-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h(\lambda) \end{pmatrix}.$$

Proof. The result is obtained by considering monomials $h(x) = x^i$, $0 \leq i < \deg(m_A)$ and adding the results. □

Proof of Theorem 1. The existence of the desired decomposition of $A = A_D + A_N$ follows from $FAF^{-1} = J = J_D + J_N$, where J_D contains all diagonal elements of J and J_N the off-diagonal elements. Since $p(A) = F^{-1}p(J)F$, we can apply Lemma 1 to each Jordan block to characterize the polynomial $p \in \mathbb{C}[x]/m_A(x)$ with $p(A) = A_D$ as the unique solution to the Hermite interpolation problem [21, p. 120]

$$p(\lambda_i) = \lambda_i,\ p^{(1)}(\lambda_i) = 0,\ \ldots,\ p^{(d_i-1)}(\lambda_i) = 0, \quad (5)$$

for $i \in \{1, \ldots, k\}$, in which λ_i denotes the i-th eigenvalue of A and d_i the size of its largest Jordan block. Thus, $A_D = p(A)$, $A_N = A - p(A)$ and $A_D A_N = A_N A_D$. □

Note that the computation of the *diagonalizable shift* $A_D = p(A)$ using Hermite interpolation only requires information about the minimal polynomial m_A of A.

Example 3. *For our running example, A in Fig. 1a, $p(x) = \frac{1}{2}x^3$, and A_D is given in Fig. 1c. The graphs associated with A and A_D are shown in Fig. 2. A_D has additional edges shown in red.*

The characteristic polynomial $\chi_A(x) = \det(A - xI)$ of A may be easier to compute than the minimal polynomial (e.g., in Matlab). Thus, we provide an alternative construction of A_D that we used in our experiments.

Lemma 2. *Let $p \in \mathbb{C}[x]/m_A(x)$ be the solution of* (5) *and $\tilde{p} \in \mathbb{C}[x]/\chi_A(x)$ be the solution of the Hermite interpolation problem*

$$\tilde{p}(\lambda_i) = \lambda_i,\ \tilde{p}^{(1)}(\lambda_i) = 0,\ \ldots,\ \tilde{p}^{(a_i-1)}(\lambda_i) = 0, \quad (6)$$

for $i \in \{1, \ldots, k\}$, in which a_i is the multiplicity of λ_i in χ_A. Then, $\tilde{p}(x) \equiv p(x) \mod m_A(x)$.

Proof. By considering the Taylor expansion of $\tilde{p}$ at λ_i, modulo $(x - \lambda_i)^{d_i}$, and applying (6) and (5) we obtain

$$\begin{aligned} \tilde{p}(x) &\equiv \tilde{p}(\lambda_i) + \sum_{l=1}^{d_i-1} \frac{\tilde{p}^{(l)}(\lambda_i)}{l!}(x - \lambda_i)^l \\ &\equiv p(\lambda_i) + \sum_{l=1}^{d_i-1} \frac{p^{(l)}(\lambda_i)}{l!}(x - \lambda_i)^l \equiv p(x) \end{aligned} \quad (7)$$

Repeating this argument for all $i \in \{1, \ldots, k\}$ and applying the Chinese remainder theorem yields the result. □

An alternative algorithm for the computation of the Jordan-Chevalley decomposition is proposed by [22].

Example 4. *Solving* (6) *for our running example (Fig. 1) yields $\tilde{p}(x) = \frac{1}{4}x^5 \equiv \frac{1}{2}x^3 = p(x) \mod x^3(x^2 - 2)$. Thus, $\tilde{p}(A) = p(A)$.*

3.2. Properties of the Diagonalizable Shift

We summarize important properties of A_D. In particular, we show that A_D generates all diagonalizable filters.

Theorem 2. (Properties of A_D) *Let A_D be the diagonalizable shift associated with A, given by Theorem 1, let $\mathcal{D} \cong \mathbb{C}[x]/m_{A_D}(x)$ denote the polynomial algebra of filters generated by A_D, and, let $FAF^{-1} = J$ be the JNF of A. Then, the following statements about A_D hold:*

P1. FA_DF^{-1} is the diagonal part J_D of J,
P2. $m_{A_D}(x) = (x - \lambda_1) \cdots (x - \lambda_k)$ and
P3. $\mathcal{D} = \{H \in \mathcal{A} : FHF^{-1}$ is diagonal$\}$.

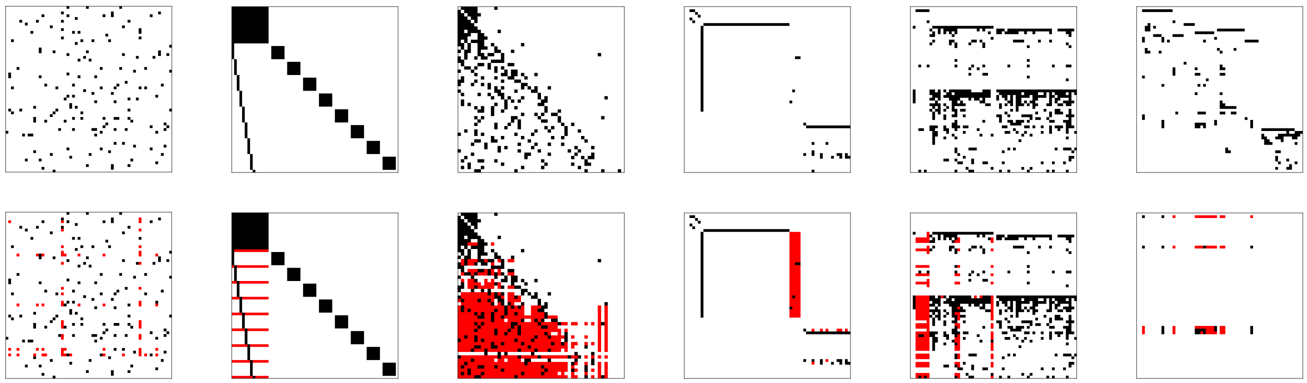
Proof. P1 holds by construction (see Theorem 1), and P2 follows from $m_{A_D}(A_D) = F m_{A_D}(J_D) F^{-1}$ combined with P1.

It remains to prove P3: Obviously, each filter in $\mathcal{D}$ has a diagonal frequency response. Thus, we only need to show that each polynomial $h \in \mathbb{C}[x]/m_A(x)$ with diagonal frequency response $h(J)$ is an element of $\mathbb{C}[x]/m_{A_D}(x) \cong \mathcal{D}$. The mapping from a filter to its frequency response $h \mapsto h(J)$ is a isomorphism, therefore, it suffices to show the existence of a polynomial $r \in \mathbb{C}[x]/m_{A_D}(x)$ with $r(\lambda_i) = h(\lambda_i)$, for $i \in \{1, \ldots, k\}$. We have $\deg(r) \leq k-1$, thus, r is the unique Lagrange interpolant for the k constraints [21, p. 119]. □

In particular, P3 means that, for an element of a Jordan subspace $\mathbf{s} \in S_{ij}$ and a filter $H = h(A_D) \in \mathcal{D}$, we have $H\mathbf{s} = h(\lambda_i)\mathbf{s}$. In [8] such filters are referred to as alias-free.

4. EXPERIMENTS

We compute and compare A_D to A for several synthetic and real-world graphs. The basis question is how A_D, again interpreted as graph differs from the original graph given by A.



(a) Erdös-Renyi (b) Pagerank (c) Barabasi-Albert (d) Stanford's web (e) Wiki-Vote (f) Arxiv HEP-PH

Fig. 3: The first row contains a selection of adjacency matrices, i.e., graph shifts, A. The second row the associated diagonalizable shifts A_D, again interpreted as graphs. Entries in A_D (i.e., edges) that are not present in A are shown in red. Weights are not shown. (a–c) are synthetic graphs, (d–f) are sub-graphs of real-world graphs.

This comparison is done in Fig. 3 for various graphs. In each case, entries (i.e., edges) in A_D not present in A are marked red. Weights in A_D are not shown.

Synthetic Graphs. We consider the following graph models:

Erdös-Renyi: In Erdös-Renyi random graphs [23] each edge is sampled independently with equal probability. For $|V| = 60$ and an edge creation probability of 0.07 about half of the randomly generated graphs are not diagonalizable. Fig. 3a shows one example where A has one Jordan block of size 3, four of size 2, and the rest of size 1.

Pagerank: Pagerank graphs [24] were used by Google to obtain rankings of websites. Each website is a vertex and there is a weighted directed edge between two vertices if there is a non-zero probability of users transitioning from the start site to the target site. Fig. 3b shows one example with nine Jordan blocks of size 2 and the rest of size 1.

Barabasi-Albert: The Barabasi-Albert random graph model [25] generates scale-free graphs that mimic social networks. This is achieved by successively growing a graph, where new nodes are more likely to connect to old nodes of high degree. For $|V| = 60$ and typical parameters [25], e.g., $m_0 = 5, m = 5$ or $m_0 = 10, m = 6$, these graphs are almost never diagonalizable. Fig. 3c shows one example with one Jordan block of size 9, one of size 4, two of size 3, two of size 2 and the rest of size 1.

Real-world Graphs. We consider the subgraphs corresponding to the first 60 vertices of graphs from the SNAP dataset [1].

Wikipedia adminship: In the Wikipedia adminship graph, users are represented as vertices. Users can vote for other users to become admin, these votes are modeled as directed edges. The subgraph in Fig. 3e has one Jordan block of size 4, three of size 2 and the rest of size 1.

Stanford web: In the Stanford web graph vertices represent pages from *stanford.edu* and directed edges represent hyperlinks between them. The subgraph in Fig. 3d has one Jordan block of size 3, four of size 2, and the rest of size 1.

Arxiv citations: We consider the the Arxiv-HEP-PH graph, which contains high energy physics phenomenology papers as vertices. Directed edges correspond to citations. The subgraph in Fig. 3f has two Jordan blocks of size 10, one of size 6, two of size 4, one of size 3, four of size 2 and the rest of size 1.

Summary. We observe that A_D, if again interpreted as graph, modifies a number of edges in A. This number depends on the amount and size of the nontrivial Jordan blocks of A and is often relatively small (except for Fig. 3c). This observation is intuitive as $A_D = A$ if all Jordan blocks are of size 1, and, the larger the blocks, the larger is the impact of the nilpotent part A_N.

5. CONCLUSION

The basic question underlying our work is how to have an operational GSP framework for digraphs in the case that the adjacency matrix A is not diagonalizable. Our solution used the Jordan-Chevalley decomposition to compute a diagonalizable A_D associated with A. Since A_D is a polynomial in A we stay within the framework of [4]. Further, our experiments suggested that A_D, if again interpreted as graph, is sometimes even similar to A, i.e., relatively few edges get modified. The idea now is to replace GSP with A by GSP with A_D. To show its viability several challenges remain including more exhaustive testing on graphs, scaling the computation of A_D to large graphs in a numerical stable way, and comparing existing GSP applications when run with A_D as shift instead of A.

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