SAMPLING SIGNALS ON MEET/JOIN LATTICES

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ABSTRACT

We present a novel sampling theorem, and prototypical applications, for Fourier-sparse lattice signals, i.e., data indexed by a finite semilattice. A semilattice is a partially ordered set endowed with a meet (or join) operation that returns the greatest lower bound (smallest upper bound) of two elements. Semilattices can be viewed as a special class of directed graphs with a strictly triangular adjacency matrix, which thus cannot be diagonalized. Our work does not build on prior graph signal processing (GSP) frameworks but on the recently introduced discrete-lattice signal processing (DLSP), which uses the meet as shift operator to derive convolution and Fourier transform. DLSP is fundamentally different from GSP in that it requires several generating shifts that capture the partial-order- rather than the adjacency-structure, and a diagonalizing Fourier transform is always guaranteed by algebraic lattice theory. We apply and demonstrate the utility of our novel sampling scheme in three real-world settings from computational biology, document representation, and auction design.

***Index Terms*—** Lattice signal, sampling, Fourier transform, meet, join, graph signal processing, algebraic signal processing

1. INTRODUCTION

The boom in big data processing and machine learning has created interest in generalizing signal processing (SP) to data on irregular domains. For example, traditional SP concepts such as convolutional filters are the backbone of state-of-the-art learning models such as convolutional neural networks and can be exchanged with their generalized counterparts [1, 2]. Generalized SP frameworks utilize the available structure of irregular data domains to derive concepts such as shifts, convolutions, Fourier transform and sampling/interpolation operators. For instance, graph signal processing (GSP) based on the Laplacian operator [3] utilizes results from algebraic graph theory to formulate a spectral convolution, and [4] proposes the adjacency matrix as shift operator and uses the algebraic signal processing (ASP) framework [5, 6] to derive convolution and Fourier transform.

Recently we used ASP to derive a novel SP framework, called discrete-lattice SP (DLSP), for signals indexed by finite lattices, or, more precisely, meet or join semilattices [7]. A semilattice is a partially ordered set L equipped with a meet (or join) operation. A meet maps a pair of lattice elements $a, b \in L$ to the largest element smaller than both a and b. For example, the powerset of a finite set ordered by inclusion and with the intersection as meet is a semilattice. In DLSP, the meet defines shift and convolution and results from algebraic lattice theory [8] guarantee and provide an associated Fourier transform.

DLSP can be viewed as a form of GSP for a special class of graphs, all of which have a strictly triangular adjacency matrix (after suitable ordering of vertices), which thus cannot be diagonalized. It is fundamentally different from GSP in that there is not one generating shift (such as, e.g., the adjacency matrix) but several, and they do not always operate among neighbors but in a way that captures the partial order structure of the domain.

Contributions. In this work we expand DLSP by investigating sampling. By instantiating classical sampling theory [9] we first derive a sampling theorem for signals that are sparse in the Fourier domain, similar as done in [10] for GSP. We then apply the results to three different types of lattice signals occurring in computational biology, text processing, and auction design. In each case the signals are exactly or approximately Fourier-sparse which enables sampling or compression.

In summary, our main contributions are as follows:

- We provide a novel sampling theorem associated with DLSP that enables the perfect reconstruction of (lattice-)Fourier k-sparse signals from k samples.
- We apply the result to real-world lattice signals in three different settings[1]: genotype-phenotype mappings [11], document representation based on prefix occurrence counts and preference elicitation schemes for combinatorial auctions [12].

Related work. Our work extends the lattice signal processing framework introduced in [7], which reinterprets results from algebraic lattice theory [8] through the lens of algebraic signal processing theory [5]. Lattice signal processing is a generalization of the signal processing framework for set functions proposed in [13]. The proposed sampling theorem draws inspiration from the sampling theory of graph signals [10], in which classical sampling theory is instantiated for GSP [4].

GSP can be broadly categorized into Laplacian-based GSP [3] and adjacency-based GSP [4]. Both frameworks do not straightforwardly apply to directed graphs (digraphs). The Laplacian operator used in [3] is not defined for digraphs and adjacency-based Fourier analysis [4] requires the Jordan decomposition of the adjacency matrix in this case. Reference [14] circumvents this problem by introducing a digraph Fourier basis that is minimal w.r.t. a (relaxed) digraph total variation. In contrast, DLSP [7] yields a different notion of Fourier transform for a special class of digraphs with strictly triangular adjacency matrices.

Lattice signals occur in various research fields, such as simplicial complexes [15, 16, 17], hypergraphs [18, 19], set functions [20, 21, 22, 23], natural language processing [24], genetics and molecular biology [11, 25, 26]. Further, several well known lattices like formal concept-, permutation- or partition lattices [27, 28, 29] might be associated with signals, e.g., statistics or fitness criteria.

[1] Sample implementation: https://github.com/chrislybaer/dlsp-sampling

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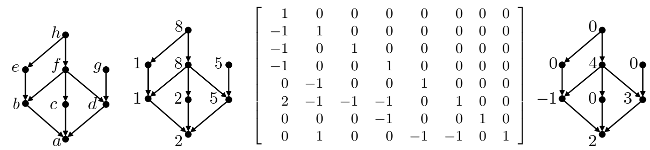


(a) L (b) $\mathbf{s}$ (c) F (d) $\hat{\mathbf{s}} = F\mathbf{s}$

Fig. 1: Meet-semilattice L, example lattice signal $\mathbf{s}$, lattice Fourier transform F, and $\hat{\mathbf{s}}$. The rows and columns of F are indexed by the lattice elements in alphabetical order.

2. BACKGROUND

In this section we provide background on the signal processing framework for lattice signals introduced in [7].

2.1. Lattice theory

We first introduce needed key concepts from lattice theory. For a more complete introduction see [8].

Lattices. We consider finite sets L. L is *partially ordered* if it permits a binary relation $\leq$ that satisfies for all $a, b, c \in L$ the following properties: (1) reflexivity: $a \leq a$, (2) anti-symmetry: $a \leq b$ and $b \leq a$ implies $a = b$, and (3) transitivity: $a \leq b$ and $b \leq c$ implies $a \leq c$. We write $a < b$ if $a \leq b$ and $a \neq b$.

A *meet-semilattice* is a partially ordered set L in which every pair $a, b \in L$ has a unique greatest lower bound $a \wedge b$. This means, $a \wedge b \leq a$, $a \wedge b \leq b$ and every c with $c \leq a$ and $c \leq b$ satisfies $c \leq a \wedge b$. The meet operation $\wedge$ will allow us to define a shift operation for lattice signals later.

An element $c \in L$ is called *meet-irreducible* if it cannot be written as the meet of two other elements, i.e., for all $a, b \in L \setminus \{c\}$ we have $c \neq a \wedge b$.

An element $b \in L$ is said to *cover* $a \in L$ iff $a < b$ and there is no $c \in L$ in between, i.e., with $a < c < b$.

Visualization. A meet-semilattice can be visualized by a directed graph $G = (L, E)$ capturing the cover-relation. Formally, $E = \{(b, a) \mid b \text{ covers } a\}$. The graph is typically drawn such that smaller elements are below larger ones. The meet $\wedge$ of two vertices $a, b \in L$ can then be visually determined. Not every directed graph defines a lattice (e.g., the directed circle does not).

Examples. Fig. 1a depicts the graph for an example lattice with meet-irreducible elements c, e, f, g, h. In L, e.g., $e \wedge f = b$, $e \wedge b = b$ and $b \wedge g = a$.

Another example of a meet-semilattice is the powerset of a finite set $N = \{x_1, \ldots, x_n\}$ with the partial order $\subseteq$ and $\wedge = \cap$ (intersection).

Finally, we note that analogously one can consider in the above a join operation $\vee$ that returns the smallest upper bound.

2.2. Discrete Lattice Signal Processing

Discrete lattice signal processing (DLSP) [7] provides basic SP concepts including convolution and Fourier transform for signals indexed by lattices. The key idea is to use the meet operation to define a shift operation and then derive all SP concepts as explained in [5]. For simplicity, we refer to meet-semilattices simply as lattices.

Lattice signal. A lattice signal maps each lattice element to a real number $s : L \to \mathbb{R} : a \mapsto s_a$ and can be represented by the $|L|$-dimensional vector $\mathbf{s} = (s_a)_{a \in L}$. We order the entries of $\mathbf{s}$ topologically sorted, i.e., larger indices come after smaller ones. In Fig. 1a the alphabetic order achieves this. Accordingly, the signal in Fig. 1b is ordered as $\mathbf{s} = (2, 1, 2, 5, 1, 8, 5, 8)^T$.

Lattice shift. The algebraic signal processing theory [5] shows that all basic SP concepts can be derived from a suitable notion of shift operation. In DLSP, the shift definition is obtained from the meet operation. Formally, for $q \in L$, the associated shift by q is defined by the linear mapping T_q

$$T_q \mathbf{s} = (s_{a \wedge q})_{a \in L}. \tag{1}$$

All possible lattice shifts are generated by those with meet-irreducible elements of L. Note that this contrasts with standard discrete-time SP and graph SP, in which there is one generating shift of which the others are polynomials. Also note that the shift operation on the lattice graph operates differently than the graph shift.

Convolution. The associated notion of convolution consists of linear shift-invariant mappings. These correspond to lattice filters $\mathbf{h}$ operating as

$$\mathbf{h} * \mathbf{s} = \Big(\sum_{q \in L} h_q s_{a \wedge q} \Big)_{a \in L}. \tag{2}$$

Shift invariance (i.e., for all $q \in L$ we have $\mathbf{h} * T_q \mathbf{s} = T_q(\mathbf{h} * \mathbf{s})$) follows from the commutativity of the meet operation.

Fourier transform. The Fourier transform $\hat{\mathbf{s}} = F\mathbf{s}$ provides an eigendecomposition w.r.t. all filters, or equivalently, diagonalizes all shifts T_q. Lattice theory [7, 8] guarantees existence and the exact form as $F = (\mu(x, y))_{y, x \in L}$, where μ denotes the Möbius function that can be computed recursively [30, 31]:

$$\begin{aligned} \mu(x, x) &= 1 && \text{for } x \in L \\ \mu(x, y) &= -\sum_{x \leq z < y} \mu(x, z) && \text{else.} \end{aligned} \tag{3}$$

The inverse of F is $F^{-1} = (\iota_{y \leq x})_{x, y \in L}$, with $\iota_{y \leq x} = 1$ if $y \leq x$ and $\iota_{y \leq x} = 0$ otherwise. Its columns are the eigenvectors (pure frequencies) of all filters. Fig. 1c shows F for the lattice in Fig. 1a. Due to the ordering, F is always lower triangular and thus not orthogonal.

As a consequence, the spectrum is also indexed by the lattice L and thus inherits its partial order. This implies a notion of (partially ordered) low and high *frequencies*: low frequencies are the columns of F^{-1} indexed by small lattice elements and high frequencies are the columns indexed by large ones. Fig. 1d shows the *spectrum* $\hat{\mathbf{s}}$ of the signal $\mathbf{s}$ in Fig. 1b.

The special case of powerset lattices was considered in [13]. An analogous version of DLSP can be derived for join-semilattices.

Relation to graph signal processing. DLSP can be viewed as an SP framework for the special class of directed graphs associated with lattices. However, there are several generating shifts (meet-irreducible elements) and not one (e.g., the adjacency matrix [4]), and thus filters, viewed as polynomials, are multivariate. Further, lattice theory guarantees a Fourier transform that diagonalizes all shifts and filters. The adjacency matrix of lattice graphs cannot be diagonalized as it is strictly triangular (up to an ordering of the vertices). Intuitively, DLSP captures the partial-order structure and not the adjacency structure.

3. SAMPLING

We now propose a novel sampling theorem for lattice signals that allows perfect reconstruction of k-sparse lattice signals using k samples. A lattice signal $\mathbf{s}$ is k-sparse if its Fourier support satisfies $|\text{supp}(\hat{\mathbf{s}})| = |\{b \in L : \hat{s}_b \neq 0\}| = k$.

$$\begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&0\\ 0&0&1&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}^T \cdot \begin{bmatrix} 2\\1\\2\\5\\1\\8\\5\\8 \end{bmatrix} = \begin{bmatrix} 2\\1\\5\\8 \end{bmatrix}$$

(a) $P_B\mathbf{s} = \mathbf{s}_B$

$$\begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 1&0&0&0\\ 0&0&1&0\\ 0&1&0&0\\ 0&0&0&1\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix} \cdot \begin{bmatrix} 2\\1\\5\\8 \end{bmatrix} = \begin{bmatrix} 2\\1\\2\\5\\1\\8\\5\\8 \end{bmatrix}$$

(b) $\mathbf{s} = I_B\mathbf{s}_B$

Fig. 2: (a) Sampling and (b) interpolation of the Fourier-sparse lattice signal shown in Fig. 1. Vectors and matrix columns are indexed by the elements of the lattice in Fig. 1a, sorted alphabetically.

In the following, we consider a k-sparse $\mathbf{s}$ with $\text{supp}(\hat{\mathbf{s}}) = \{b_1, \ldots, b_k\} = B$. Following the paradigm of classical sampling theory [9], we are looking for a linear sampling operator P_A that reduces the signal $\mathbf{s}$ to k samples ($|A| = k$) such that there exists a linear interpolation operator I_A that allows for perfect recovery of $\mathbf{s}$ from these samples.

Sampling. Formally, for $A \subseteq L$ with $|A| = |\text{supp}(\hat{\mathbf{s}})| = k$, $|L| = n$, we consider linear sampling operators of the form

$$P_A : \mathbb{R}^n \to \mathbb{R}^k : \mathbf{s} \mapsto \mathbf{s}_A = (s_{a_1}, \ldots, s_{a_k}). \quad (4)$$

Interpolation. The corresponding linear interpolation operator

$$I_A : \mathbb{R}^k \to \mathbb{R}^n : \mathbf{s}_A \mapsto \mathbf{s} \quad (5)$$

exists if the sub-matrix $F^{-1}_{A,B}$, obtained by selecting the rows $A \subseteq L$ and columns $B \subseteq L$ from F^{-1}, is invertible.

In other words, finding a pair of sampling and interpolation operator for a Fourier-sparse signal with $\text{supp}(\hat{\mathbf{s}}) = B$ boils down to selecting elements $A = \{a_1, \ldots, a_k\} \subseteq L$ such that the linear system of equations

$$s_a = \sum_{b \in B} \hat{s}_b \mathbf{f}^b_a \text{ for } a \in A, \quad (6)$$

where $\mathbf{f}^b$ denotes the b-th pure frequency (= the b-th column of F^{-1}), admits a unique solution. Theorem 1 yields a sample selection criterion and the respective interpolation operator.

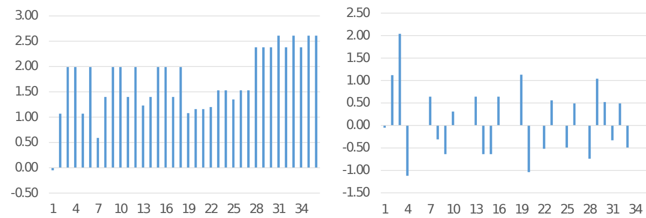
Theorem 1. (Lattice sampling) *Let $\mathbf{s}$ be a lattice signal on L with supp($\hat{\mathbf{s}}$) $= \{b_1, \ldots, b_k\} = B$. Then $\mathbf{s}$ can be reconstructed from the samples $\mathbf{s}_B = P_B\mathbf{s}$. Namely, $\mathbf{s} = I_B\mathbf{s}_B$ with $I_B = F^{-1}_{L,B}(F^{-1}_{B,B})^{-1}$. The matrix $F^{-1}_{L,B}$ is the sub-matrix of F^{-1} obtained by selecting the rows L and columns B.*

Proof. Because of the Fourier-sparsity of $\mathbf{s}$, we have $\mathbf{s} = F^{-1}_{L,B}\hat{\mathbf{s}}_B$. Applying the sampling operator P_B to both sides yields $\mathbf{s}_B = F^{-1}_{B,B}\hat{\mathbf{s}}_B$. Therefore, proving our claim amounts to showing that $F^{-1}_{B,B}$ is of full rank.

When choosing an order of the elements $a \in L$ that agrees with the partial order defined by $\leq$, the matrix F^{-1} becomes a lower triangular matrix with $F^{-1}_{aa} = \iota_{a \leq a} = 1$. Consequently, sub-matrices of the form $F^{-1}_{B,B}$ have always full rank as desired. □

In the previous example (Fig. 1), $\mathbf{s}$ is 4-sparse with $B = \{a, b, d, f\}$. Fig. 2 applies Theorem 1 to sample and reconstruct $\mathbf{s}$.

In practice, we might not have access to the entire lattice signal. Instead, sampling may amount to querying an evaluation oracle [21], taking measurements [11] or eliciting preferences [12]. Additionally, real-world signals might not be exactly sparse. However, if they are approximately sparse, Theorem 1 can still be applied to obtain a sampling-based approximation.



(a) $\mathbf{s}$ (b) $\hat{\mathbf{s}}$

Fig. 3: (a) Genotype-lattice signal and (b) its spectrum.

4. EXAMPLE APPLICATIONS

In the following we explore various examples of lattice signals and investigate whether they admit a sparse representation. In all our signal and spectrum plots the lattice elements are topologically sorted as described in Sec. 2.2.

4.1. Genotype Lattice Signals

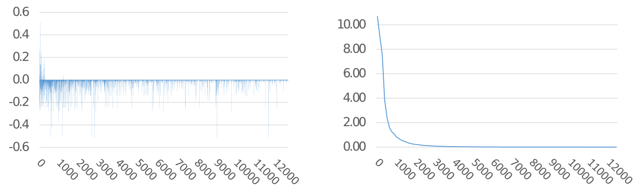
In genetics and molecular biology, genotype-phenotype mappings are studied to better understand pathogens (viruses, bacteria, parasites, cancer cells) [11, 25], e.g., to derive HIV treatment schemes that take the resistance of a particular HIV gene to various drugs into account [32]. A gene is a subsequence of a DNA sequence occurring at a particular position and a genotype is a variant of a gene caused by mutations at possibly multiple positions of the gene. A genotype-phenotype mapping associates genotypes with real numbers, e.g., indicating their levels of resistance to a certain drug. References [11, 26] utilize advances in statistics and machine learning to infer such genotype-phenotype mappings from possibly incomplete or noisy data [33, 34]. The shared underlying assumption of these approaches is that genotypes form a sublattice of the powerset lattice of mutations compatible with the possible mutation order. In other words, genotype-phenotype mappings are lattice signals.

Lattice. Formally, let $M = \{m_1, \ldots, m_l\}$ be the set of mutations for one gene. For $m_1, m_2 \in M$, we write $m_1 \leq m_2$ if mutation m_1 *must* occur before mutation m_2 can occur. A genotype $g \subseteq M$ is defined as a subset of mutations that obeys the constraints among mutations, i.e., $m_1 \in g$ implies $m_2 \in g$ for all $m_2 \leq m_1$. The set of all valid genotypes is thus a sub-semilattice L of the powerset lattice 2^M with $\wedge = \cap$.

Signal. As signals we consider genotype-phenotype mappings that associate virus genotypes with their resistances to a certain set of drugs.

Experiment. We compute the lattice Fourier transforms of the two reported estimated genotype-phenotype lattice mappings in [11, Appendix Table 4 & Table 7]. These lattice signals are of length 28 and 36, respectively.

The first maps each HIV RT genotype to its resistance against the nucleotide RT inhibitor zidovudine. RT genotypes are composed of up to seven amino acid substitutions in the RT gene. 11 out of 28 Fourier coefficients are zero. The second lattice signal (Fig. 3a) maps each HIV PR genotype to its resistance against the PR inhibitor indinavir. PR genotypes are composed of up to six amino acid substitutions in the PR gene. 12 of the 36 Fourier coefficients are zero (Fig. 3b). In both cases, the signals could thus be losslessly sampled



(a) $\hat{\mathbf{s}}$ (b) $\|\mathbf{s} - \mathbf{s}'\|/\|\mathbf{s}\|$

Fig. 4: (a) Spectrum of normalized prefix-count signal $\|\mathbf{s}\| = 1$ and (b) relative reconstruction error using the k largest spectral components.

using Theorem 1. Next, we consider two much larger examples of lattice signals.

4.2. Word Lattice Signals

We consider lattice signals derived from text. There are multiple ways to define a partial order over words [35] and some of them allow for the definition of a meet operation. Here we consider the prefix order as an example.

Lattice. Let Σ be a finite alphabet, Σ^* the set of finite words using letters from that alphabet and for a word $u = u_1u_2 \cdots u_l \in \Sigma^*$ we denote its length by $|u| = l$. The prefix order is defined as follows: For $u, v \in \Sigma^*$ we have $u \leq v$ iff $|u| \leq |v|$ and $u_i \leq v_i$ for $i \in \{1, \dots, |u|\}$. Thus, the greatest lower bound of two words with respect to $\leq$ is their longest common prefix: $u \wedge v = u_1u_2 \cdots u_p$ where $u_1 = v_1, \dots, u_p = v_p$ and $u_{p+1} \neq v_{p+1}$.

Given a document $D = (u^{(1)}, \dots, u^{(m)}) \in (\Sigma^*)^m$, we consider the smallest sub-semilattice L of the prefix-semilattice Σ^* containing the document (i.e., $\{u^{(1)}, \dots, u^{(m)}\} \subseteq L$).

Signal. As a signal we assign to every $a \in L$ how often it occurs as prefix in the document D. For example, for $a =$ the, we count every occurrence of the words the, their, then, etc.

Experiment. We construct a signal $\mathbf{s}$ from the book "Die Kritik der reinen Vernunft" by Immanuel Kant from project Gutenberg[2]. The book contains $9,952$ unique words, which expands to $12,636$ prefixes after we include all meets to obtain a semilattice. Fig. 4a shows the spectrum of the signal. It is approximately sparse, meaning that the absolute values of about 90% of the spectral coefficients are smaller than 0.01. We reconstruct the signal as $\mathbf{s}'$ using the k largest spectral components $|\hat{s}_b|\|\mathbf{f}^{(b)}\|$ (i.e., from the associated samples given by Theorem 1). Fig. 4b shows the reconstruction error as a function of k. For example, an error of 10% can be achieved with less than 2500 samples.

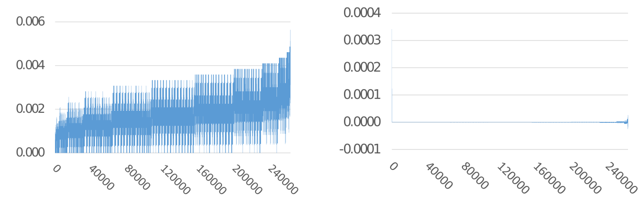
4.3. Combinatorial Auction Valuations

Combinatorial auctions are an electronic market design paradigm concerned with the auctioning of a set of goods [12] sold in bundles to a set of bidders. The goal is to find an allocation of the goods to the bidders that maximizes the social welfare. An important example are auctions for bands of the electromagnetic spectrum [22].

Lattice. Formally, a bundle $b \subseteq M$ is a subset of a finite set of goods $M = \{g_1, \dots, g_m\}$. The bundles form a powerset lattice 2^M with $\wedge = \cap$.

Signal. Each individual bidder is modeled as a valuation function. A valuation function $s : 2^M \to \mathbb{R}^{\geq 0}$ captures the preferences of a bidder, i.e., it associates each bundle with its bidder-specific value. Therefore, each bidder defines a lattice signal $\mathbf{s}$.

[2] https://www.gutenberg.org/ebooks/6343



(a) $\mathbf{s}$ (b) $\hat{\mathbf{s}}$

Fig. 5: (a) Example valuation function and (b) its spectrum.

The social welfare of an allocation of M to the bidders is the sum of the respective values, but the value functions are unknown to the auctioneer. Therefore, the auctioneer tries to estimate the bidders' valuation functions by querying values for specific bundles, i.e., by sampling. Thus, Theorem 1 yields a sampling scheme for bidders with Fourier-sparse $\mathbf{s}$. The challenge is to find the support of the spectrum.

Experiment. For the auction design, it is unfeasible to obtain complete valuation functions for real world bidders, thus, it is common practice to make use of simulated bidders. In our experiment, we used the Global Synergy Value Model (GSVM) [36] to generate valuation functions for seven bidders $\mathbf{s}^{(1)}, \dots, \mathbf{s}^{(7)}$ in a world with 18 goods $M = \{g_1, \dots, g_{18}\}$ using the spectrum auction test suite [22]. In the default GSVM setup there are one national and six regional bidders. The goods are arranged equidistantly on two circles, twelve on the national circle and the remaining six on the regional one. National bidders are interested in all goods on the national circle and regional bidders in four goods on the national circle and two on the regional one. The strengths of the synergy effects depend on the angular distances on the circles.

The spectrum of all $\mathbf{s}^{(i)}$ is very sparse with the largest values at $\hat{\mathbf{s}}_B^{(i)}$ for the 172 subsets $B \subseteq M$ with $|B| \leq 2$, i.e., the functions are band-limited. An example is shown in Fig. 5. In Fig. 5b the first 172 spectral coefficients amount to the barely visible peak at zero. The subsets are ordered by cardinality. Thus, using the first 172 samples we can approximately reconstruct the 7 functions. The relative reconstruction error is $\|\mathbf{s}^{(i)} - \mathbf{s}'^{(i)}\|/\|\mathbf{s}^{(i)}\| = 4.31 \cdot 10^{-6} \pm 1.75 \cdot 10^{-6}$.

As a consequence, under the assumption that a bidder is GSVM, an auctioneer could sample all goods and pairs of goods to obtain a reasonable estimate of the entire valuation function.

5. CONCLUSIONS

We expanded the recently introduced discrete-lattice signal processing to investigate sampling and include a sampling theorem. Since the lattice Fourier transform is triangular (if the lattice is topologically sorted), it takes a somewhat simple form in that, for perfect reconstruction, the sampling locations are equal to the sparsity support in the Fourier domain. As prototypical examples we considered three very different classes of lattice signals and showed that all were sparse or approximately sparse in the Fourier domain, thus allowing for practical sampling. The results show that DLSP provides meaningful new tools, different from graph SP, to process a relevant class of real-world signals.

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