Algebraic Signal Processing Theory

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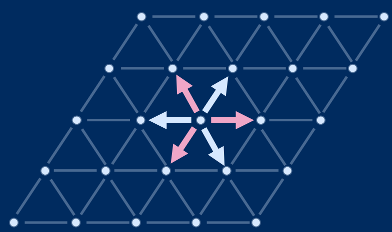


$$(\mathcal{A}, \mathcal{M}, \Phi)$$

Theory

Picture: http://www.christianch.ch/images/andere_gefahren.png

Preliminaries

- *ASP = algebraic signal processing theory*

- *Algebra:* **theory of groups, rings, and fields**
- **Scope of ASP:** ***linear signal processing***
- **This talk: Focus on the *discrete* case**

Signal processing concepts	Infinite time	Finite time	Infinite space	Finite space	*ASP:* Generic case	Other/ new models
	z-transform	Finite z-transform	C-transform	Finite C-transform	Φ	
Set of signals	Laurent series in z^{-n}	Polynomials in z^{-n}	Series in C_n	Polynomials in C_n	$\mathcal{M}$	
Set of filters	Laurent series in z^{-n}	Polynomials in z^{-n}	Series in T_n	Polynomials in T_n	$\mathcal{A}$	
Fourier transform	DTFT	DFT	DSFT	DCTs/DSTs	$\mathcal{F}$	
Convolution						
Spectrum					*Derivation*	
Frequency response						
Fast algorithms						
Filter banks						
<many others>						

Big Picture

- **Key concept in ASP:**

Signal model: $(\mathcal{A}, \mathcal{M}, \Phi)$

algebra of filters — signal ***module*** — associated "z-transform"

- **Examples: Infinite and finite time or space, many others**
- **Signal model defined: all other concepts follow**

Signal model →

- Signal
- Filter
- "z-transform"
- Spectrum
- Fourier transform
- Frequency response
- Fast algorithms
- *… many other concepts*

Big Picture

- **Key concept in ASP:**

Signal model: $(\mathcal{A}, \mathcal{M}, \Phi)$

algebra of filters — signal ***module*** — associated "z-transform"

- **Examples: Infinite and finite time or space, many others**
- **Signal model defined: all other concepts follow**

Signal model →

- Element of module
- Element of algebra
- Mapping into module
- Irreducible submodules
- Decomposition into irred. submodules
- Irreducible representation
- Stepwise induction
- … many other concepts

Organization

- *The algebraic structure underlying linear signal processing*
- **From shift to signal model: Time and space**
- **From infinite to finite signal models**
- **Fast algorithms**
- **Conclusions**

Algebraic Structure of Signal and Filter Space

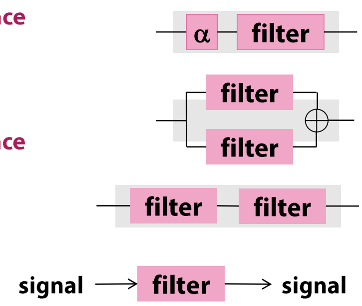
- **Signal space, available operations:**
  - $\alpha \cdot$ signal = signal
  - signal + signal = signal

  } **vector space**

- **Filter space, available operations:**
  - $\alpha \cdot$ filter = filter
  - filter + filter = filter
  - filter $\cdot$ filter = filter

  } **vector space** (first two) } **ring** (last two)

- **Filters operate on signals:**
  - filter $\cdot$ signal = signal



Set of filters = an algebra $\mathcal{A}$

Set of signals = an $\mathcal{A}$-module $\mathcal{M}$

(Algebraic) Signal Model

- **Signals arise as sequences of numbers** $(s_n)_{n\in I} \in \mathbb{C} \times \mathbb{C} \times \cdots = \mathbb{C}^I$
- **Need to assign module and algebra**
- **Example infinite discrete time:** $(s_n)_{n\in\mathbb{Z}}$

z-transform: $\Phi: (s_n)_{n\in\mathbb{Z}} \to \sum s_n z^{-n} \in \mathcal{M}$

$\mathcal{M} = \{\sum s_n z^{-n}\}$ $\quad \mathcal{A} = \{\sum h_k z^{-k}\}$

- ***Signal model (definition):*** $(\mathcal{A}, \mathcal{M}, \Phi)$

$\mathcal{A}$ **algebra of filters**
$\mathcal{M}$ **an $\mathcal{A}$-module of signals**
Φ **linear mapping $\mathbb{C}^I \to \mathcal{M}$**

Algebras Occurring in SP: Shift-Invariance

- **What is the shift?**
  - A special filter x (=z^{-1}) = an element of $\mathcal{A}$
  - Filters expressible as polynomials/series in x

shift(s) = generator(s) of $\mathcal{A}$

- **Shift-invariance:** $x \cdot h = h \cdot x$ for all $h \in \mathcal{A}$

signal model $(\mathcal{A}, \mathcal{M}, \Phi)$ is shift-invariant $\iff$ $\mathcal{A}$ is commutative

- **Shift-invariant + finite-dimensional (+ one shift only):**

$\mathcal{A} = \mathbb{C}[x]/p(x)$ **is a polynomial algebra**

Example: Finite Time Model and DFT

- **Finite signals:** $(s_0, s_1, \dots, s_{n-1})$ $\dim(\mathcal{M}),\ \dim(\mathcal{A}) < \infty$
- **Signal model:** $\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/(x^n - 1)$

signals $s(x) = \sum_{i=0}^{n-1} s_i x^i \in \mathcal{M}$ *filters* $h(x) = \sum_{k=0}^{n-1} h_k x^k \in \mathcal{A}$

filtering = cyclic convolution $h(x) \cdot s(x) \bmod x^n - 1$

finite z-transform $\Phi: \quad \mathbb{C}^n \to \mathcal{M}$

$$(s_0, s_1, \dots, s_{n-1}) \mapsto \sum s_i x^i \in \mathcal{M}$$

- **Spectrum/Fourier transform: Chinese remainder theorem**

$$\begin{aligned} \mathcal{F}: \ \mathbb{C}[x]/(x^n - 1) &\to \mathbb{C}[x]/(x - \omega_n^0) \oplus \dots \oplus \mathbb{C}[x]/(x - \omega_n^{n-1}) \\ s(x) &\mapsto (s(\omega_n^0), \dots, s(\omega_n^{n-1})) \end{aligned}$$

$\mathcal{F} = \mathrm{DFT}_n$

Summary so far

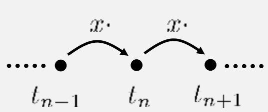
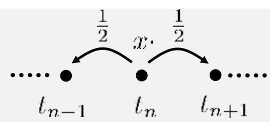
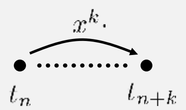
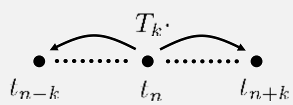
- **Signal model** $(\mathcal{A}, \mathcal{M}, \Phi)$
- **Shift-invariance:** $\mathcal{A}$ **is commutative**
  - in addition finite makes $\mathcal{A}$ a polynomial algebra
- **Infinite and finite time are special cases of signal models**

How to go beyond time?
Answer: **Derivation of signal model from shift**

shift → **signal model** $(\mathcal{A}, \mathcal{M}, \Phi)$

Organization

- The algebraic structure underlying linear signal processing
- *From shift to signal model: Time and space*
- From infinite to finite signal models
- Fast algorithms
- Conclusions

	Time	Space
	$x \cdot t_n = t_{n+1}$	$x \cdot t_n = \frac{1}{2}(t_{n-1} + t_{n+1})$
shift *(time) marks*		
k-fold shift		
realization of (time) marks	$t_0 = 1 \Rightarrow t_n = x^n$	$t_0 = 1 \Rightarrow t_n = C_n$
signals	$s = \sum s_n x^n$	$s = \sum s_n C_n$
filters	$h = \sum h_k x^k$	$h = \sum h_k T_k$

Chebyshev polynomials

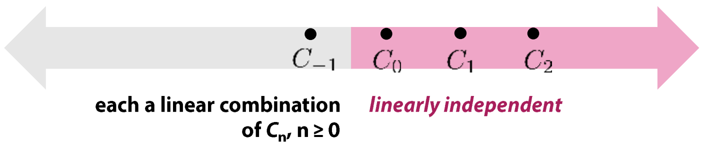
Time and Space (cont'd)

Chebyshev polynomials

- **Time: done** $\mathcal{M} = \{\sum s_n x^n\}$ $\mathcal{A} = \{\sum h_k x^k\}$

  $\Phi: \ (s_n)_{n\in\mathbb{Z}} \to \sum s_n x^n \in \mathcal{M}$ ***z-transform***

- **Space:** $\mathcal{M} = \{\sum s_n C_n\}$ $\mathcal{A} = \{\sum h_k T_k\}$



- **Signal model only for right-sided sequences:**

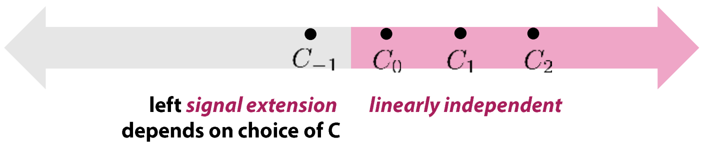
  $\Phi: \ (s_n)_{n\in\mathbb{Z}} \to \sum_{n\geq 0} s_n C_n \in \mathcal{M}$ ***C-transform***

Left Signal Extension

Chebyshev polynomials

- **Infinite space model:** $\mathcal{M} = \{\sum_{n\geq 0} s_n C_n\}$ $\mathcal{A} = \{\sum_{k\geq 0} h_k T_k\}$

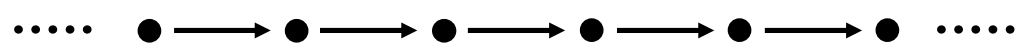
  $\Phi: \ (s_n)_{n\in\mathbb{Z}} \to \sum_{n\geq 0} s_n C_n \in \mathcal{M}$



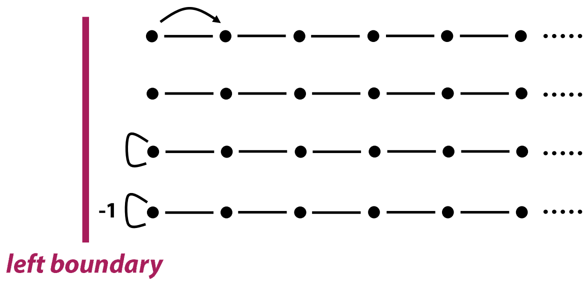
- **Simplest signal extension: *monomial*** $C_{-n} = aC_k$
- **Monomial if and only if** $C \in \{T, U, V, W\}$

Visualization

- **Infinite discrete time (z-transform)**



- **Infinite discrete space (C-transform, C = T, U, V, W)**



$$T_{-n} = T_n$$

$$U_{-n} = -U_{n-2}$$

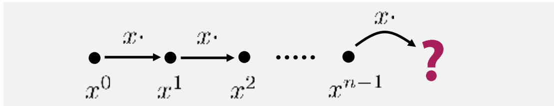
$$V_{-n} = V_{n-1}$$

$$W_{-n} = -W_{n-1}$$

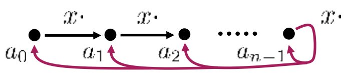
Organization

- **The algebraic structure underlying linear signal processing**
- **From shift to signal model: Time and space**
- ***From infinite to finite signal models***
- **Fast algorithms**
- **Conclusions**

Derivation: *Finite* Time Model



- **Solution: *Right boundary condition***



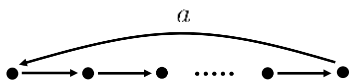
$$x^n = a_{n-1}x^{n-1} + \cdots + a_0 x^0$$

$$p(x) = x^n - a_{n-1}x^{n-1} - \cdots - a_0 x^0 = 0$$

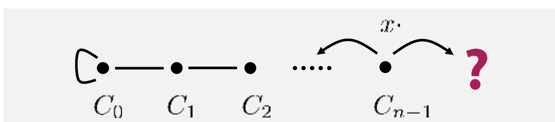
$$\mathcal{M} = \mathcal{A} = \mathbb{C}[x]/p(x)$$

- **Monomial *signal extension*:** $p(x) = x^n - a, \quad a \neq 0$
  (a = 1: finite z-transform)

- **Visualization:**



Derivation: *Finite* Space Model



$$C \in \{T, U, V, W\}$$

- **Monomial *signal extension*: For each** $C \in \{T, U, V, W\}$ **four cases**

$$\begin{aligned} C_n &= C_{n-2} \\ C_n &= 0 \\ C_n &= C_{n-1} \\ C_n &= -C_{n-1} \end{aligned}$$

- **16 finite space models** $\Longleftrightarrow$ **16 DCTs/DSTs as Fourier transforms**

16 Finite Space Models

	$s_n - s_{n-2}$	s_n	$s_n - s_{n-1}$	$s_n + s_{n-1}$	f	C
$s_{-1} = s_1$	**DCT-1** $2(x^2-1)U_{n-2}$	**DCT-3** T_n	**DCT-5** $(x-1)W_{n-1}$	**DCT-7** $(x+1)V_{n-1}$	1	T
$s_{-1} = 0$	**DST-3** $2T_n$	**DST-1** U_n	**DST-7** V_n	**DST-5** W_n	$\sin\theta$	U
$s_{-1} = s_0$	**DCT-6** $2(x-1)W_{n-1}$	**DCT-8** V_n	**DCT-2** $2(x-1)U_{n-1}$	**DCT-4** $2T_n$	$\cos\frac{1}{2}\theta$	V
$s_{-1} = -s_0$	**DST-8** $2(x+1)V_{n-1}$	**DST-6** W_n	**DST-4** $2T_n$	**DST-2** $2(x+1)U_{n-1}$	$\sin\frac{1}{2}\theta$	W

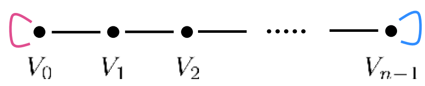
- **Example: Signal model for DCT, type 2:**

$$\mathcal{M} = \mathbb{C}[x]/(x-1)U_{n-2} = \{\sum_{i=0}^{n-1} s_i V_i\}$$

$$\mathcal{A} = \mathbb{C}[x]/(x-1)U_{n-2} = \{\sum_{k=0}^{n-1} h_k T_k\}$$

$$\Phi : \ (s_i)_{0 \geq i < n} \mapsto \sum_{i=0}^{n-1} s_i V_i$$

- **Visualization:**



Φ	$\mathcal{M}$	$\mathcal{A}$	$\mathcal{F} = \mathcal{P}_{b,\alpha}$	other $\mathcal{F}$
$\mathbf{s} \mapsto \sum s_k x^k$	$\mathbb{C}[x]/(x^n - a)$	regular	$\mathrm{DFT}_n \cdot D$	—
	$\mathbb{C}[x]/(x^n - 1)$	regular	$\mathrm{DFT}_n = \text{DFT-1}_n$	DFT-2_n
	$\mathbb{C}[x]/(x^n + 1)$	regular	DFT-3_n	DFT-4_n

Time (real): real finite z-transform — Section VI-G

Φ	$\mathcal{M}$	$\mathcal{A}$	$\mathcal{F} = \mathcal{P}_{b,\alpha}$	other $\mathcal{F}$
$\mathbf{s} \mapsto \sum s_k x^k$	$\mathbb{R}[x]/(x^n - 1)$	regular	n.a.	$\mathrm{RDFT}_n = \text{RDFT-1}_n$
	$\mathbb{R}[x]/(x^n - 1)$	regular	n.a.	RDFT-2_n
	$\mathbb{R}[x]/(x^n - 1)$	regular	n.a.	$\mathrm{DHT}_n = \text{DHT-1}_n$ (DWT-1_n)
	$\mathbb{R}[x]/(x^n - 1)$	regular	n.a.	DHT-2_n (DWT-2_n)
	$\mathbb{R}[x]/(x^n + 1)$	regular	n.a.	RDFT-3_n
	$\mathbb{R}[x]/(x^n + 1)$	regular	n.a.	RDFT-4_n
	$\mathbb{R}[x]/(x^n + 1)$	regular	n.a.	DHT-3_n (DWT-3_n)
	$\mathbb{R}[x]/(x^n + 1)$	regular	n.a.	DHT-4_n (DWT-4_n)

Space (complex/real): finite C-transform (C = T,U,V,W) — Sections VIII-B, IX-B, XI-B

Φ	$\mathcal{M}$	$\mathcal{A}$	$\mathcal{F} = \mathcal{P}_{b,\alpha}$	other $\mathcal{F}$
$\mathbf{s} \mapsto \sum s_k T_k$	$\mathbb{C}[x]/(x^2-1)U_{n-2}$	regular	$\text{DCT-1}_n = \overline{\text{DCT-1}}_n$	—
	$\mathbb{C}[x]/T_n$	regular	$\text{DCT-3}_n = \overline{\text{DCT-3}}_n$	—
	$\mathbb{C}[x]/(x-1)W_{n-1}$	regular	$\text{DCT-5}_n = \overline{\text{DCT-5}}_n$	—
	$\mathbb{C}[x]/(x+1)V_{n-1}$	regular	$\text{DCT-7}_n = \overline{\text{DCT-7}}_n$	—
	$\mathbb{C}[x]/(T_n - \cos r\pi)$	regular	$\text{DCT-3}_n(r) = \overline{\text{DCT-3}}_n(r)$	—
$\mathbf{s} \mapsto \sum s_k U_k$	$\mathbb{C}[x]/T_n$	regular	$\overline{\text{DST-3}}_n$	DST-3_n
	$\mathbb{C}[x]/U_n$	regular	$\overline{\text{DST-1}}_n$	DST-1_n
	$\mathbb{C}[x]/V_n$	regular	$\overline{\text{DCT-7}}_n$	DCT-7_n
	$\mathbb{C}[x]/W_n$	regular	$\overline{\text{DST-5}}_n$	DST-5_n
	$\mathbb{C}[x]/(T_n - \cos r\pi)$	regular	$\overline{\text{DST-3}}(r)_n$	$\text{DST-3}(r)_n$
$\mathbf{s} \mapsto \sum s_k V_k$	$\mathbb{C}[x]/(x-1)W_{n-1}$	regular	$\overline{\text{DCT-6}}_n$	DCT-6_n
	$\mathbb{C}[x]/V_n$	regular	$\overline{\text{DCT-8}}_n$	DCT-8_n
	$\mathbb{C}[x]/(x-1)U_{n-1}$	regular	$\overline{\text{DCT-2}}_n$	DCT-2_n
	$\mathbb{C}[x]/T_n$	regular	$\overline{\text{DCT-4}}_n$	DCT-4_n
	$\mathbb{C}[x]/(T_n - \cos r\pi)$	regular	$\overline{\text{DCT-4}}(r)_n$	$\text{DCT-4}(r)_n$
$\mathbf{s} \mapsto \sum s_k W_k$	$\mathbb{C}[x]/(x+1)V_{n-1}$	regular	$\overline{\text{DST-8}}_n$	DST-8_n
	$\mathbb{C}[x]/W_n$	regular	$\overline{\text{DST-6}}_n$	DST-6_n
	$\mathbb{C}[x]/T_n$	regular	$\overline{\text{DCT-4}}_n$	DCT-4_n
	$\mathbb{C}[x]/(x+1)U_{n-1}$	regular	$\overline{\text{DST-2}}_n$	DST-2_n
	$\mathbb{C}[x]/(T_n - \cos r\pi)$	regular	$\overline{\text{DST-4}}(r)_n$	$\text{DST-4}(r)_n$
$\mathbf{s} \mapsto \sum s_k x^k$	$\mathbb{C}[x]/(x^n - 1)$	$\langle (x^{-1}+x)/2 \rangle$	n.a.	$\mathrm{RDFT}_n = \text{RDFT-1}_n$
	$\mathbb{C}[x]/(x^n - 1)$	$\langle (x^{-1}+x)/2 \rangle$	n.a.	RDFT-2_n
	$\mathbb{C}[x]/(x^n - 1)$	$\langle (x^{-1}+x)/2 \rangle$	n.a.	$\mathrm{DHT}_n = \text{DHT-1}_n$
	$\mathbb{C}[x]/(x^n - 1)$	$\langle (x^{-1}+x)/2 \rangle$	n.a.	DHT-2_n
	$\mathbb{C}[x]/(x^n + 1)$	$\langle (x^{-1}+x)/2 \rangle$	n.a.	RDFT-3_n
	$\mathbb{C}[x]/(x^n + 1)$	$\langle (x^{-1}+x)/2 \rangle$	n.a.	RDFT-4_n
	$\mathbb{C}[x]/(x^n + 1)$	$\langle (x^{-1}+x)/2 \rangle$	n.a.	DHT-3_n
	$\mathbb{C}[x]/(x^n + 1)$	$\langle (x^{-1}+x)/2 \rangle$	n.a.	DHT-4_n

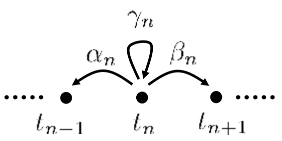
1D Trigonometric Transforms

- Signal models for all existing (and some newly introduced) trigonometric transforms (~30)
- Explains all existing trigonometric transforms
- Gives for each transform associated "z-transform", filters, etc.

source: "Algebraic Theory of Signal Processing," Arxiv

More Exotic 1-D Model

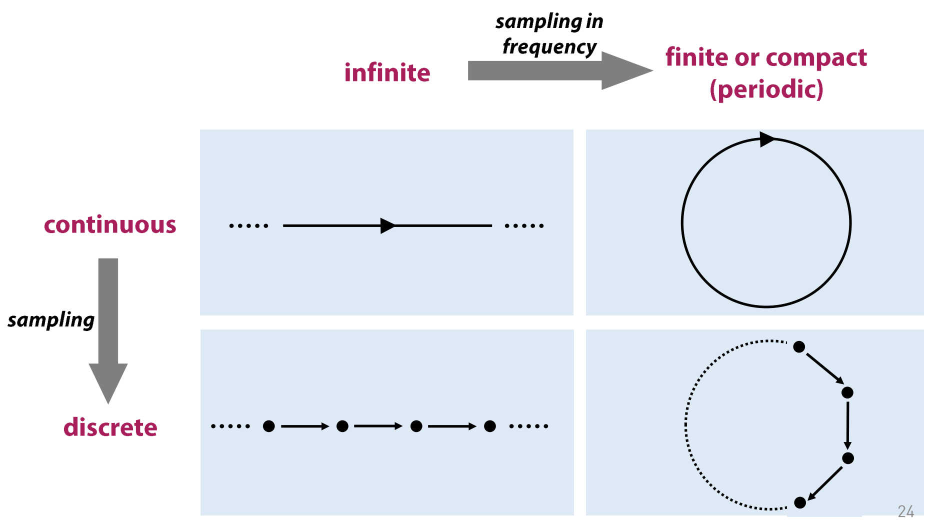
- Generic next neighbor shift



- Space *variant* but shift invariant
- Connects to orthogonal polynomials
- Applications?

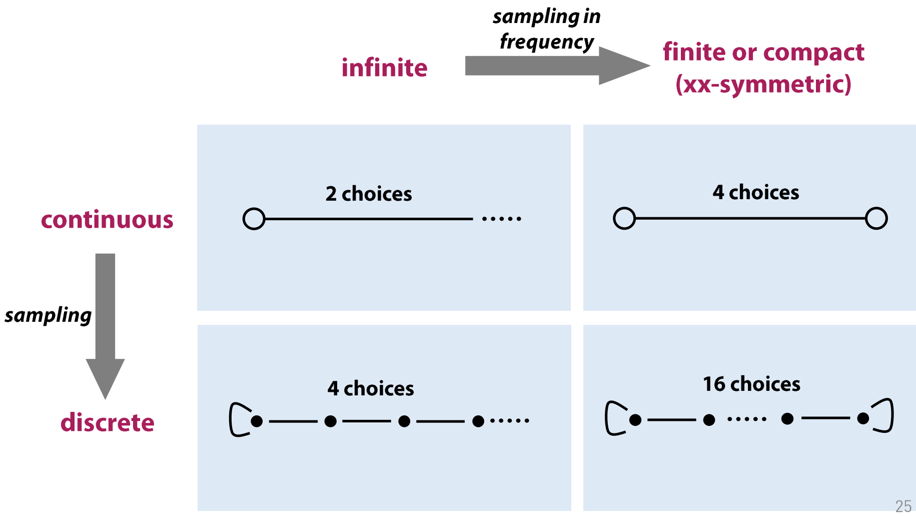
Top-Down: 1-D Time (Directed) Models

$h * s = \int s(\tau)h(t-\tau)d\tau$

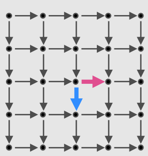
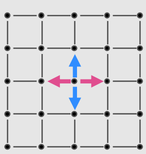


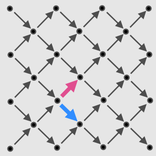
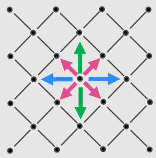
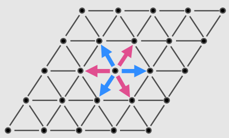
Top-Down: 1-D Space (Undirected) Models

$h * s = \int s(\tau)\frac{1}{2}(h(t-\tau)+h(t+\tau))d\tau$



Finite Signal Models in Two Dimensions

Visualization (without b.c.)	Signal Model $\mathcal{A} = \mathcal{M}$	Fourier Transform
 *2-D time, separable*	$\mathbb{C}[x,y]/\langle x^n-1, y^n-1\rangle$ **time shifts: x, y**	$\mathrm{DFT}_n \otimes \mathrm{DFT}_n$
 *2-D space, separable*	$\mathbb{C}[x,y]/\langle T_n(x), T_n(y)\rangle$ **space shifts: x, y**	$\mathrm{DCT}_n \otimes \mathrm{DCT}_n$ **16 types**

 *time, nonseparable*	$\mathbb{C}[u,v]/\langle u^n - 1, u^{n/2} - v^{n/2}\rangle$ **time shifts: u, v**	$\mathrm{DDQT}_{n\times n/2}$
 *space, nonseparable*	$\mathbb{C}[u,v,w]/\langle T_{n/2}(u), T_{n/2}(v), 4w^2 - (u+1)(v+1)\rangle$ **space shifts: u, v, w**	$\mathrm{DQT}_{n\times n/2}$
 *space, nonseparable*	$\mathbb{C}[x,y]/\langle C_n(x,y), \overline{C}_n(x,y)\rangle$ **space shifts: u, v**	$\mathrm{DTT}_{n\times n}$

Organization

- **The algebraic structure underlying linear signal processing**
- **From shift to signal model: Time and space**
- **From infinite to finite signal models**
- ***Fast algorithms***
- **Conclusions**

Fast Algorithms: Two Schools

$\omega_n = e^{-2\pi j/n}$

DFT

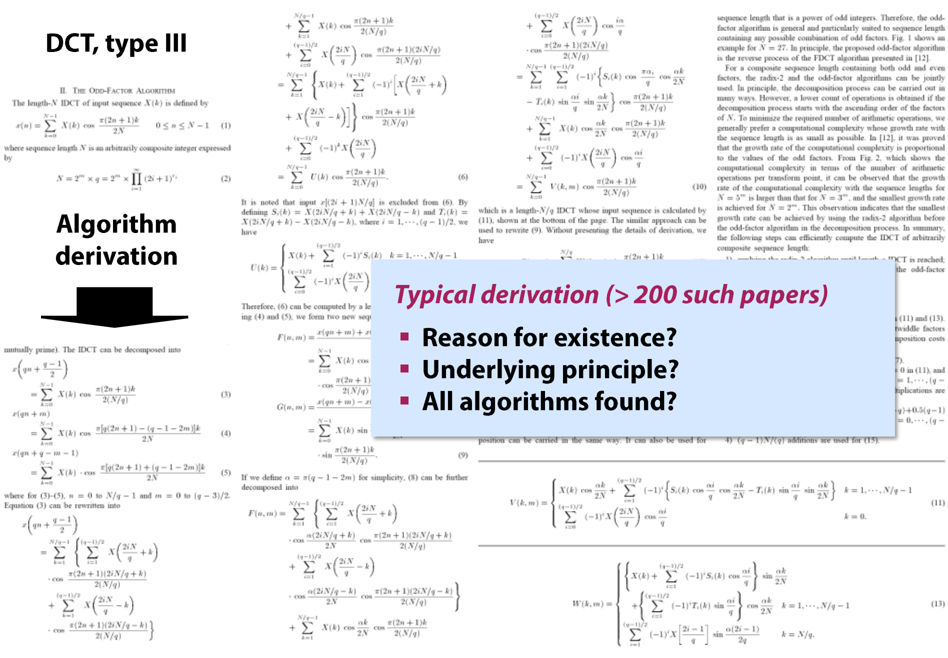
$$y_k = \sum_{\ell=0}^{n-1} \omega_n^{k\ell} s_\ell \qquad\qquad y = \mathrm{DFT}_n \cdot s, \quad \mathrm{DFT}_n = [\omega_n^{k,\ell}]_{0\le k,\ell<n}$$

Cooley-Tukey FFT (general radix)

$$y_{n_2 j_1 + j_2} = \sum_{k_1=0}^{n_1-1} (\omega_n^{j_2 k_1}) \left(\sum_{k_2=0}^{n_2-1} s_{n_1 k_2 + k_1} \omega^{j_2 k_2} \right) \omega^{j_1 k_1} \qquad \mathrm{DFT}_n = L_{n_2}^n (I_{n_1} \otimes \mathrm{DFT}_{n_2}) T_{n_1}^n (\mathrm{DFT}_{n_1} \otimes I_{n_2})$$

DCT, type III

Algorithm derivation



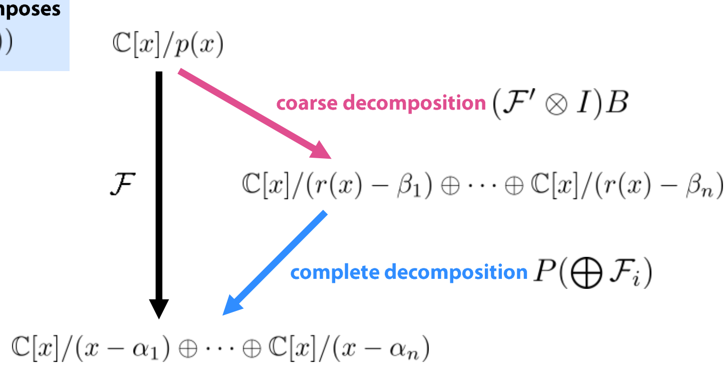
Typical derivation (> 200 such papers)

- **Reason for existence?**
- **Underlying principle?**
- **All algorithms found?**

G. Bi "Fast Algorithms for the Type-III DCT of Composite Sequence Lengths" IEEE Trans. SP 47(7) 1999

Cooley-Tukey FFT Type Algorithms

assume p decomposes
$p(x) = q(r(x))$



Example: $x^n - 1 = (x^m)^k - 1$ **yields Cooley-Tukey FFT**

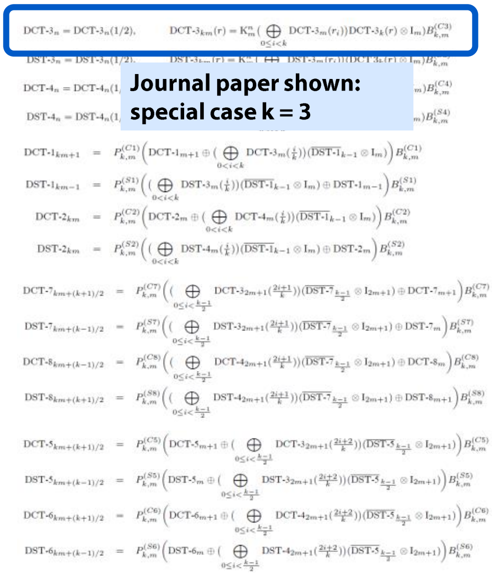
$$\mathrm{DFT}_n = L^n_{n_2}(I_{n_1} \otimes \mathrm{DFT}_{n_2}) T^n_{n_1} (\mathrm{DFT}_{n_1} \otimes I_{n_2})$$

Application to DCTs/DSTs

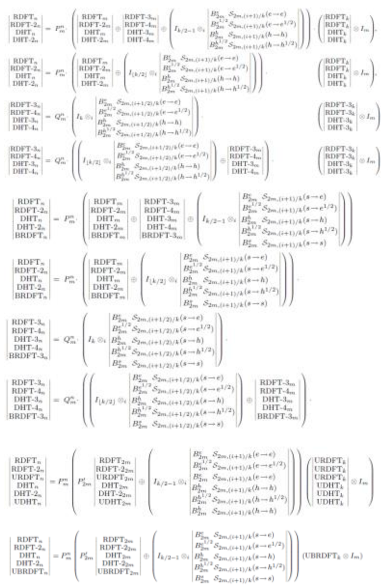
- **Decomposition properties of Chebyshev polynomials**

$$\begin{aligned}
&T_{km} = T_k(T_m)\\
&U_{km-1} = U_{m-1} \cdot U_{k-1}(T_m)\\
&V_{(k-1)/2+km} = V_m \cdot V_{(k-1)/2}(T_{2m+1})\\
&W_{(k-1)/2+km} = W_m \cdot W_{(k-1)/2}(T_{2m+1})\\
&T_{km+m/2} = T_{m/2} \cdot V_k(T_m)\\
&U_{km+m/2-1} = U_{m/2-1} \cdot W_k(T_m)
\end{aligned}$$

Induced algorithms
DCTs/DSTs



Real DFTs/DHTs



<many more>

Algebraic Theory of Transform Algorithms

- **Consolidates the area**
- **Few general principles account for practically all existing algorithms**
  - General Cooley-Tukey type algorithms
  - General prime-factor type algorithms
  - General Rader type algorithms
- **Derivation is greatly simplified**
- **Many (dozens) new algorithms discovered**
- **Applicable to existing and novel linear transforms**
  - DCTs/DSTs, MDCTs, RDFT, DHT, DQT, DTT, …

Organization

- The algebraic structure underlying linear signal processing
- From shift to signal model: Time and space
- From infinite to finite signal models
- Fast algorithms
- *Conclusions*

Markus Püschel and José M. F. Moura
Algebraic Signal Processing Theory: Foundation and 1-D Time
IEEE Transactions on Signal Processing, Vol. 56, No. 8, pp. 3572-3585, 2008

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Algebraic Signal Processing Theory: 1-D Space
IEEE Transactions on Signal Processing, Vol. 56, No. 8, pp. 3586-3599, 2008

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Algebraic Signal Processing Theory: Cooley-Tukey Type Algorithms for DCTs and DSTs
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Algebraic Signal Processing Theory: Cooley-Tukey Type Algorithms for Real DFTs
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Jelena Kovacevic and Markus Püschel
Algebraic Signal Processing Theory: Sampling for Infinite and Finite 1-D Space
IEEE Transactions on Signal Processing, Vol. 58, No. 1, pp. 242-257, 2010

Markus Püschel and Martin Rötteler
Algebraic Signal Processing Theory: Cooley-Tukey Type Algorithms on the 2-D Spatial Hexagonal Lattice
Applicable Algebra in Engineering, Communication and Computing, special issue on "The memory of Thomas Beth", Vol. 19, No. 3, pp. 259-292, 2008

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Aliaksai Sandryhaila, Jelena Kovacevic and Markus Püschel
Algebraic Signal Processing Theory: Cooley-Tukey Type Algorithms for Polynomial Transforms Based on Induction
SIAM Journal on Matrix Analysis and Applications, Vol. 32, No. 2, pp. 364-384, 2011

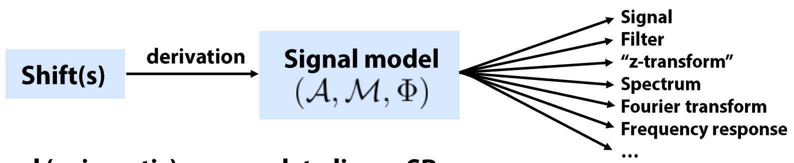
Aliaksei Sandryhaila, Jelena Kovacevic and Markus Püschel
Algebraic Signal Processing Theory: 1-D Nearest-Neighbor Models
IEEE Transactions on Signal Processing, Vol. 60, No. 5, pp. 2247-2259, 2012

Related Work on Algebraic Methods in SP

- ***Fourier analysis/Fourier transforms on groups G*** **(Beth, Rockmore, Clausen, Maslen, Healy, Terras, …)**
  - In the algebraic theory the special case $\mathcal{A} = \mathcal{M} = \mathbb{C}[G]$
  - If G non-commutative, necessarily non-shift-invariant
  - Algebraic theory provides associated filters etc., ties to SP concepts
- ***Algebraic methods to derive DFT algorithms*** **(Nicholson, Winograd, Nussbaumer, Auslander, Feig, Burrus, …)**
  - Recognizes algebra/module for DFT, but only used for deriving algorithms
- ***Origin of this work***
  - Beth (84), Minkwitz (93), Egner/Püschel (97/98)
  - Helpful hints: Steidl (93), Moura/Bruno (98), Strang (99)
- ***Algebraic systems theory*** **(Kalman, Basile/Marro, Wonham/Morse, Willems/Mitter, Fuhrmann, Fliess, …)**
  - Focuses on infinite discrete time; different type of questions

Algebraic Signal Processing Theory: Conclusions

- **Signal model: One concept instantiating different SP methods**



- **General (axiomatic) approach to linear SP**
- **Deeper insight into SP**
- **Applications to date:**
  - New signal models (non-separable 2-D)
  - Comprehensive theory of fast algorithms

`SMART project: www.ece.cmu.edu/~smart`

Chebyshev Polynomials

back1
back2
back3

- **Defining three-term recurrence:**

  initial: $C_0 = 1, \quad C_1 = ax + b$

  $$C_{n+1} = 2xC_n + C_{n-1} \quad \Longleftrightarrow \quad xC_n = \tfrac{1}{2}(C_{n+1} + C_{n-1})$$

- **Special cases:**

C	$\dots$	$n=-1$	$n=0$	$n=1$	$n=2$	$\dots$
T	$\dots$	x	1	x	$2x^2-1$	$\dots$
U	$\dots$	0	1	$2x$	$4x^2-1$	$\dots$
V	$\dots$	1	1	$2x-1$	$4x^2-2x-1$	$\dots$
W	$\dots$	-1	1	$2x+1$	$4x^2+2x-1$	$\dots$

n ≥ 0 →

- **Closed forms:** $\cos\theta = x$

$$T_n = \cos n\theta \qquad U_n = \frac{\sin(n+1)\theta)}{\sin\theta} \qquad V_n = \frac{\cos(n+\frac{1}{2})\theta)}{\cos\frac{1}{2}\theta} \qquad W_n = \frac{\sin(n+\frac{1}{2})\theta)}{\sin\frac{1}{2}\theta}$$