Wiener Filter on Meet/Join Lattices

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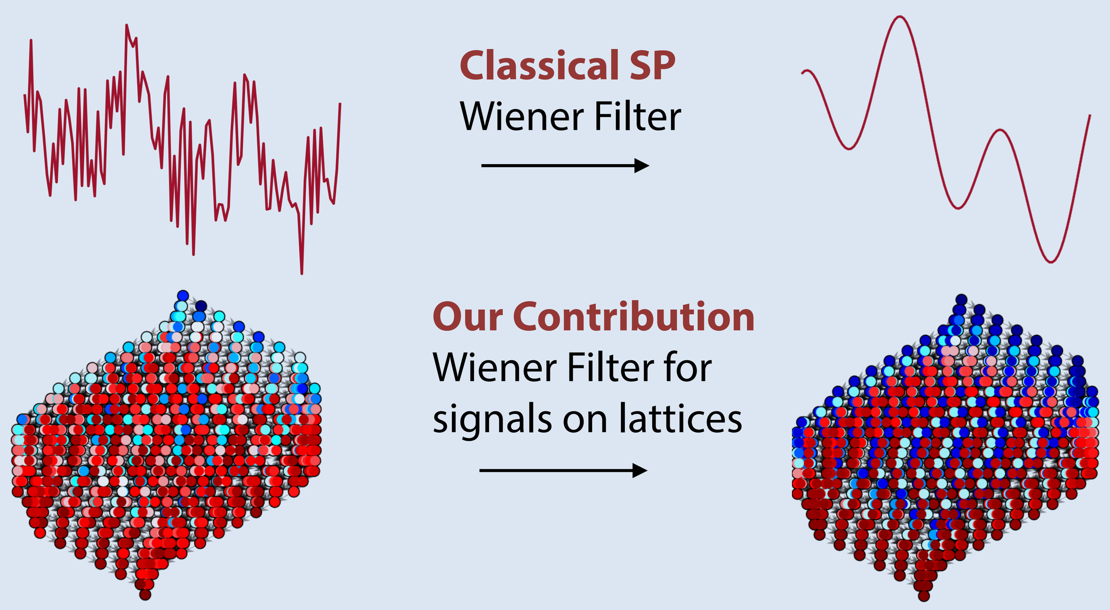
Goal



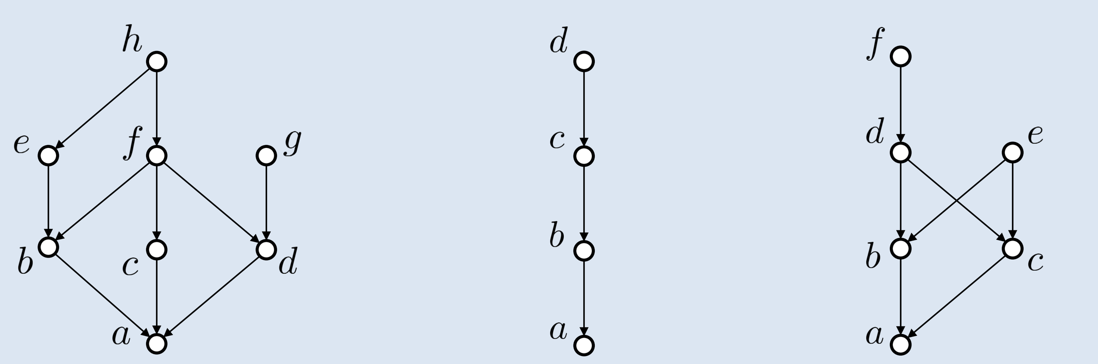
Lattices

Meet Semilattice

- Finite set L with **partial order** $\leq$
- **Meet operation:** $a \wedge b$ (greatest lower bound of a and b)
- b **covers** a: $a < b$, no $x : a < x < b$

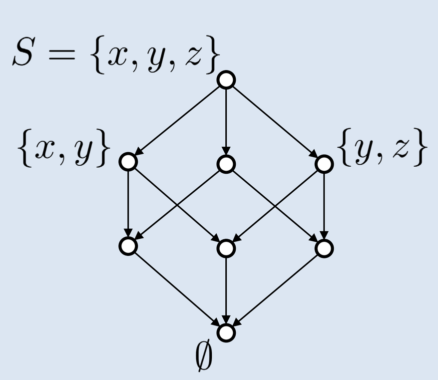
Cover Graph Representation of Lattices

- Nodes: L
- Edges: $(b, a) \Leftrightarrow b$ covers a



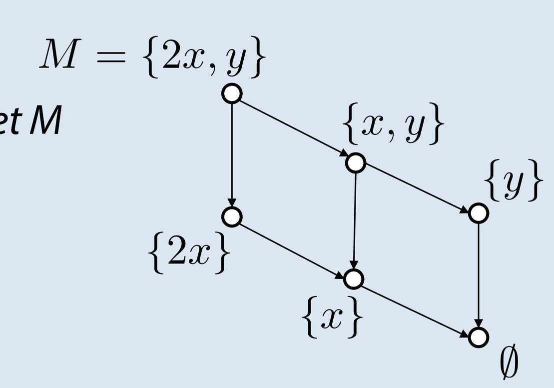
Powerset Lattice

Elements: *Subsets of given set S*
Partial Order: $\subseteq$
Meet: $\cap$



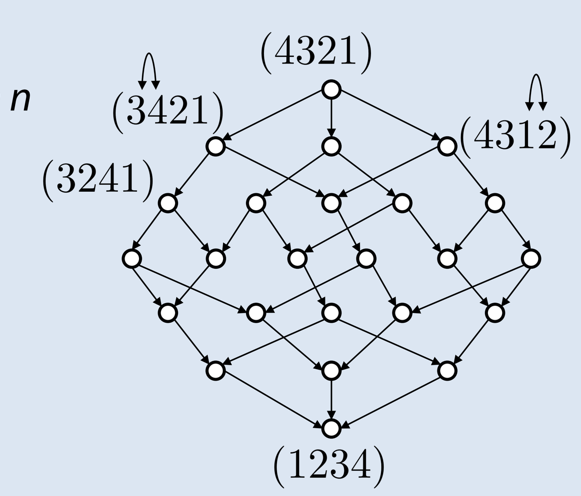
Multiset Lattice

Elements: *Submultisets of multiset M*
Partial Order: $\subseteq$
Meet: $\cap$



Permutation Lattice

Elements: *Permutations of length n*
b covers a: $(b_1, \ldots, b_n) = (a_1, \ldots, a_{i+1}, a_i, \ldots, a_n)$, $a_i < a_{i+1}$
Partial Order and Meet: *Derived from cover graph*

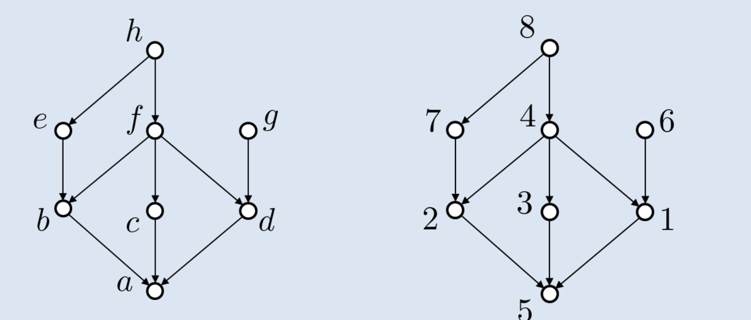


Grätzer, *Lattice Theory: Foundation*, 2011

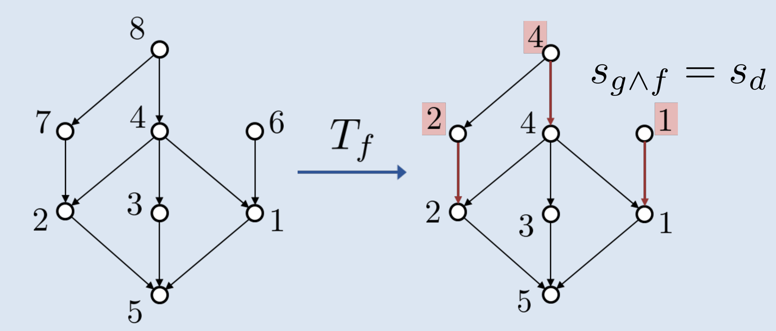
Lattice Signal Processing

Derivation: Algebraic Signal Processing
Shift ⟶ Convolution ⟶ Fourier Transform

Lattice Signal
$\mathbf{s} = (s_x)_{x \in L} \in \mathbb{R}^n$



Shift(s) by $q \in L$
$T_q \mathbf{s} = (s_{x \wedge q})_{x \in L}$

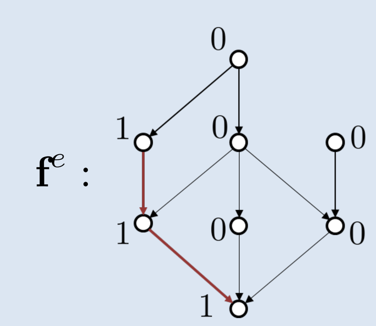


Convolution is shift-invariant

$$\mathbf{h} * \mathbf{s} = \Big(\sum_{q \in L} h_q s_{x \wedge q}\Big)_{x \in L}$$

Fourier Basis Vectors

$\mathbf{f}^y = (\iota_{y \leq x})_{x \in L}, \quad y \in L$
└ *characteristic function*



Fourier Transform diagonalizes all shifts and filters

$$\widehat{s}_y = \sum_{x \leq y} \mu(x, y) s_x \qquad \mu(x, x) = 1, \text{for } x \in L$$

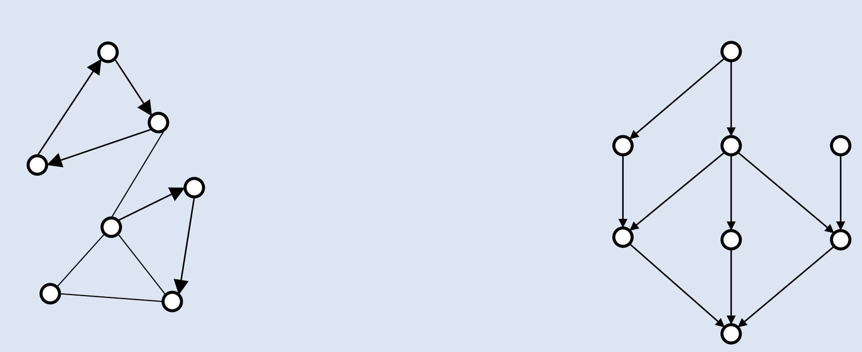
$$s_x = \sum_{y \leq x} \widehat{s}_y \qquad \mu(x, y) = -\sum_{x \leq z < y} \mu(x, z), \quad \text{for } x \neq y$$

Fourier transform matrix for above lattice

$$\mathrm{DLT} = [\mu(x, y)]_{y, x \in L} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & -1 & -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 & 1 \end{bmatrix}$$

A Discrete Signal Processing Framework for Meet/Join Lattices, ICASSP 2019
Discrete Signal Processing on Meet/Join Lattices, arXiv, submitted for publication

Lattice DSP vs. Graph DSP



Graph DSP:

Shift captures adjacency structure

One generating shift
(adjacency or Laplacian)

Shift not always diagonalizable
(digraphs)

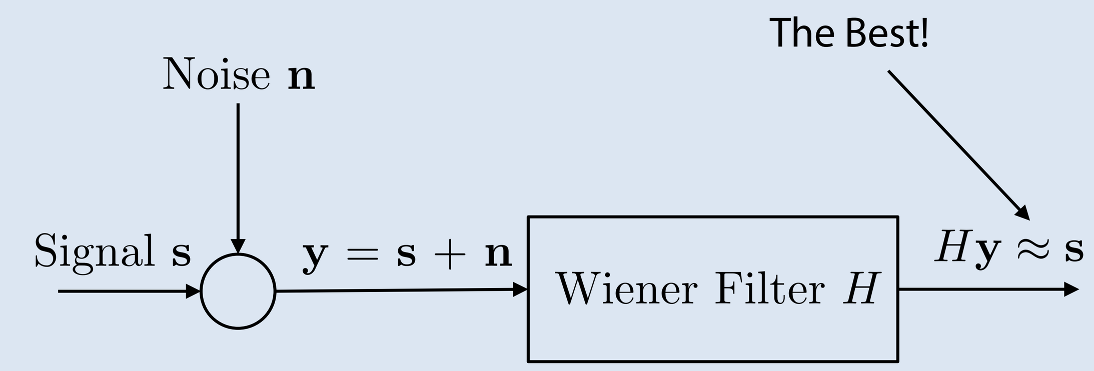
Lattice DSP:

Shifts capture partial order structure

Several generating shifts
(one per "maximal" element)

Shifts always diagonalizable

Wiener Filter on Lattices



Key idea: *(For Graphs: Gavili, Zhang Trans. Signal Proc. 2017)*

Replace Shifts with Energy-Preserving Shift

$$T_{\mathrm{ep}} = \mathrm{DLT}^{-1} \cdot \Lambda_{\mathrm{ep}} \cdot \mathrm{DLT}$$

with $\Lambda_{\mathrm{ep}} = \mathrm{diag}(\exp(2\pi\, i\, k / |L|) \mid k = 0, \ldots, |L| - 1)$

Preserves energy

$$\|T_{\mathrm{ep}}\mathbf{s}\|_2 = \|\widehat{\mathbf{s}}\|_2$$

Every other shift representable

$$T_q = p_q(T_{\mathrm{ep}})$$

Wiener Filter

$$H = \sum_{k=0}^{N} h_k T_{\mathrm{ep}}^k$$

Filter Coefficients $\mathbf{h} = (h_0, \ldots, h_N)$

$$\min_{\mathbf{h}} \|H\mathbf{y} - \mathbf{s}\|_2^2$$

Closed-form solution

$$B^H B \mathbf{h} = B^H \mathbf{s}$$

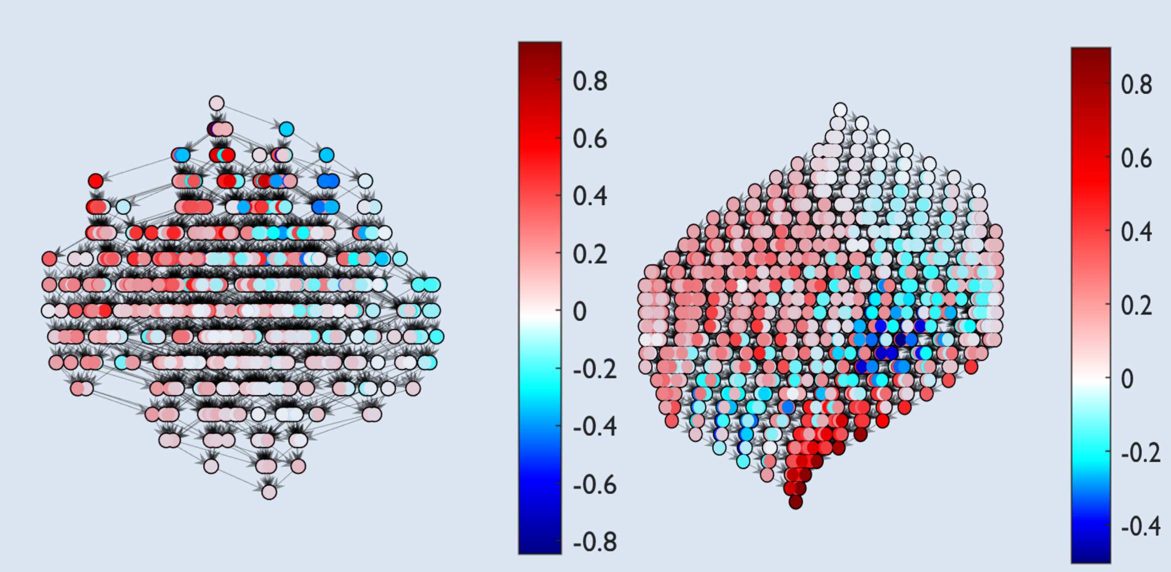
$$\text{with } B = [\mathbf{y} \; T_{\mathrm{ep}}\mathbf{y} \; \ldots \; T_{\mathrm{ep}}^N \mathbf{y}]$$

Lattice White Noise

$\mathbf{n} = \mathrm{DLT}^{-1}\, \hat{\mathbf{n}}$
└ *Gaussian noise vector*

Q: Why?

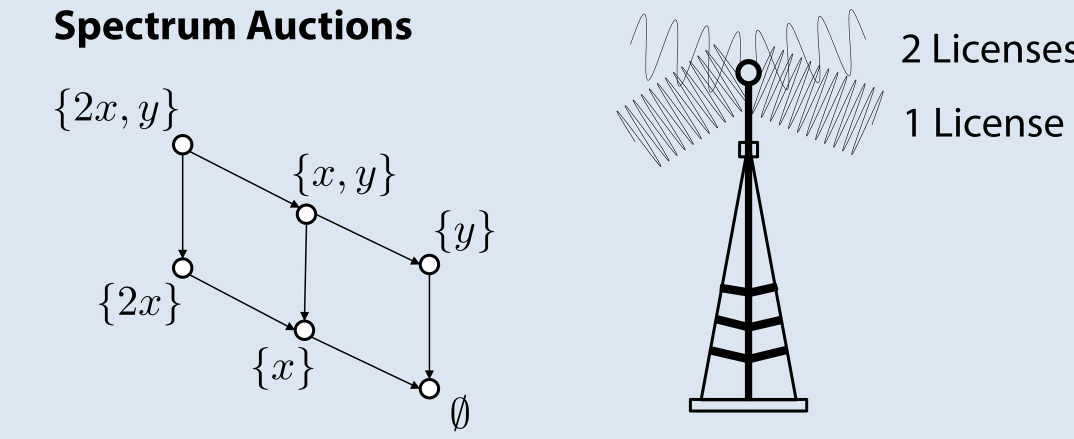
A: Lattice Fourier transform is **not** orthogonal. Thus $\mathbf{n}$ is not Gaussian.



Two examples of lattice white noise on different lattices

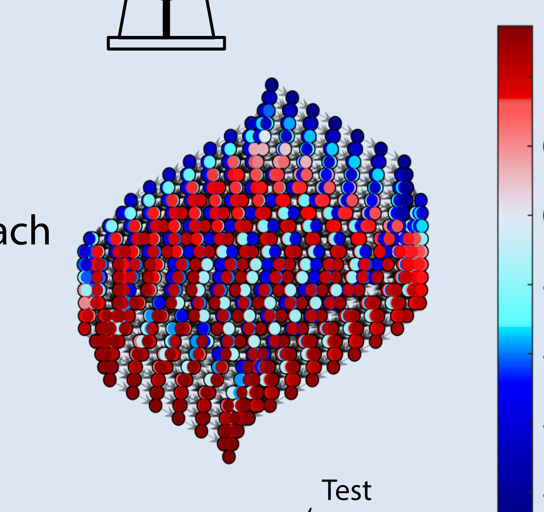
Applications

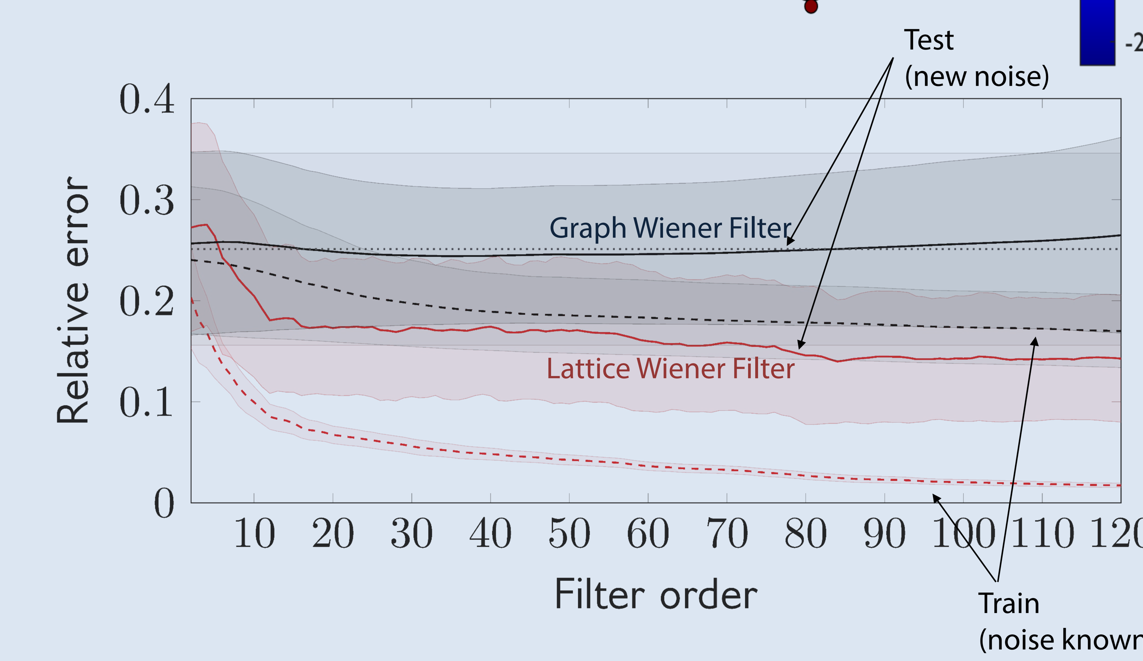
Spectrum Auctions



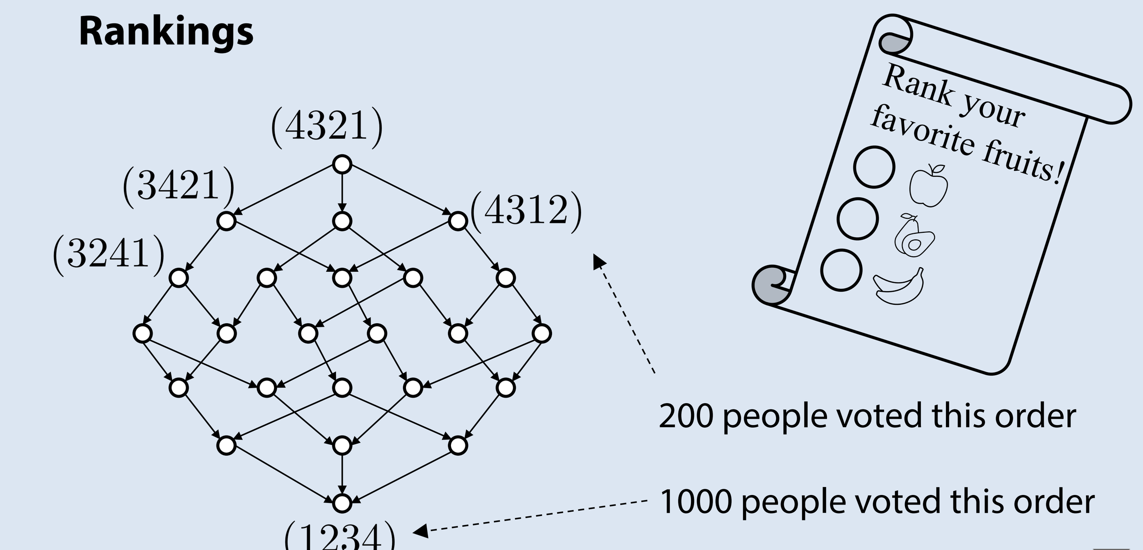
Multiset of licenses = lattice

Bidder = signal of values for each submultiset of licenses





Rankings



Rankings = lattice of permutations of choices

Poll = signal of how many people voted for each permutation